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Spaces $H_\phi^+(\Omega)$ and $H_\phi(\Omega)$

فضاءات $H_\phi(\Omega)$ و $H_\phi^+(\Omega)$

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Abstract

Let Ω be an open connected subset of the complex plane **C**, $H(\Omega)$ the space of all analytic functions in Ω , and ϕ is a modulus function such that $\phi(|\ f\ |)$ is subharmonic in Ω for all f $H(\Omega)$ $\in$. In this paper we define $H_\phi(\Omega)$ to be the space of all f $H(\Omega)$ $\in$ such that $\phi(|\ f\ |)$ has a harmonic majorant and $H_\phi^+(\Omega)$ is the space of all f $H_\phi(\Omega)$ $\in$ such that $\phi(|\ f\ |)$ has a quasi-bounded harmonic majorant.

This extends the special cases $H^p(\Omega)$ when $0 < p \le 1$ $\phi(x) = x^p$, ,and $N(\Omega)$ and $N^+(\Omega)$ when $\phi(x) = \log(1+x)$. It also extends N^p from $p \geq 1$ to p > 0 where $\phi(x) = (\log(1+x))^p$ and Ω is the open unit disc D and includes N_p where $\phi(x) = \log(1+x^p),\ 0 < p < 1$. We show that $H_\phi(\Omega)$ is a complete metric space and $H_\phi^+(\Omega)$ is an F-space which generalizes the special case Ω = D.Also we show that many results for= H_ϕ $H_\phi(D)$ and $H_\phi^+(D)$ = H_ϕ^+ carry over to $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ **.** Different characterizations of H_ϕ and H_ϕ^+ are given and it is shown that $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ can be identified with closed subspaces of H_ϕ when ϕ is a strictly increasing unbounded modulus function. This result is used to give an other proof of the completeness of $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$. When Ω is finitely connected a factorization theorem for functions in $H_\phi(\Omega)$

and $H^+_\phi(\Omega)$ is given. Also, a necessary and sufficient integrability condition for functions $f \in H^+_\phi(\Omega)$ as well as a formula for the least harmonic majorant of $\phi(|f|)$ are given.

ملخص

لتكن Ω مجموعه مفتوحه ومترابطه وجزئيه من المستوي العقدي **C**، وليكن $H(\Omega)$ فضاء الدوال التحليليه في Ω و ϕ داله مطلقه القيمه بحيث أن $\phi(|f|)$ داله توافقيه جزئيا في Ω لكل $f \in H(\Omega)$. في هذا البحث نعرف $H_\phi(\Omega)$ على أنه فضاء جميع الدوال $f \in H(\Omega)$ بحيث أن $\phi(|f|)$ يكون لها داله توافقيه تحدها من أعلى ونعرف $H^+_\phi(\Omega)$ على أنه فضاء جميع الدوال $f \in H_\phi(\Omega)$ بحيث أن $\phi(|f|)$ يكون لها داله توافقيه شبه محدوده وتحدها من أعلى.هذا يعمم الحاله الخاصه $H^p(\Omega)$، عندما، $0 < p \le 1$، $\phi(x) = x^p$ و $N(\Omega)$ و $N^+(\Omega)$ عندما $\phi(x) = \log(1+x)$. سنثبت، ان شاء الله،أن $H_\phi(\Omega)$ فضاءا متريا كاملا و $H^+_\phi(\Omega)$ فضاء F مما يعمم الحاله الخاصه التي تكون فيها Ω قرص الوحده المفتوح **D**. أيضا سنثبت، ان شاء الله، أن كثيرا من النتائج في الحاله الخاصه $H_\phi(D) = H_\phi$ و $H^+_\phi(D) = H^+_\phi$ يمكن أن تنسحب على $H_\phi(\Omega)$ و $H^+_\phi(\Omega)$. أيضا سنقدم، ان شاء الله، أوصافا متنوعه ل H_ϕ و H^+_ϕ إذ يمكن وصف $H_\phi(\Omega)$ و $H^+_\phi(\Omega)$ بواسطه فضاءات مغلقه وجزئيه من H_ϕ عندما تكون ϕ داله مطلقه القيمه ومتزايده بصرامه وغير محدوده.سنستخدم هذه النتيجه لاعطاء برهانا اخرا لكون $H_\phi(\Omega)$ و $H^+_\phi(\Omega)$ فضاءات كامله.عندما تكون Ω محدوده الترابط سنقدم ، ان شاء الله ،نظريه لتحليل الدوال في $H_\phi(\Omega)$ و $H^+_\phi(\Omega)$ وقاعده لأصغر داله توافقيه وتحد من أعلى $\phi(|f|)$ لكل $f \in H^+_\phi(\Omega)$. أيضا سنقدم شرطا تكامليا كافيا ولازما للدوال في $H^+_\phi(\Omega)$.

1. Introduction and Preliminaries

If ϕ is a real-valued function on $[0, \infty)$ such that ϕ is increasing, subadditive , $\phi(x) = 0$ iff $x = 0$,and continuous at zero from the right (hence uniformly continuous on $[0, \infty)$),then ϕ is called a modulus function. Examples of modulus functions are $x^p, 0 < p \le 1$,and $\log(1+x)$. We note that if ϕ is a modulus function, then so is c ϕ where c> 0.Also, the composition of two modulus functions is a modulus function.

Let T = ∂D be the boundary of the open unit disc D in the complex plane **C** and H(D) the space of analytic functions in D .Let $H^{+}(D)$ be the set of all functions $f \in H(D)$ such that

$$\lim_{r\to 1} f(re^{i\theta}) = f^{*}(e^{i\theta}) \text{ exists a.e.}\sigma$$

where σ is the normalized Lebesgue measure on T. f^{*} is called the radial

limit of f .When there is no ambiguity we denote the function f and its radial limit by f .Throughout this paper we assume that ϕ is a modulus function such that $\phi(|f|)$ is subharmonic in D for all $f \in H(D)$ .We define the Hardy-Orlicz spaces $H_{\phi}(D) = H_{\phi}$ and $H_{\phi}^{+}(D) = H_{\phi}^{+}$ by

$$H_{\phi} = \{ f \in H(D) : \sup_{0\le r<1} \int_{T} \phi(|f_r|) d\sigma < \infty \}$$

and

$$H_{\phi}^{+} = \{ f \in H^{+}(D) : \sup_{0\le r<1} \int_{T} \phi(|f_r(z)|) d\sigma(z) = \int_{T} \phi(|f(z)|) d\sigma(z) < \infty \}$$

where $f_r(z) = f(rz), z \in T$.

For each $f \in H_{\phi}(D)$,define the quasi-norm of f by

$$\| f \|_{\phi} = \sup_{0\le r<1} \int_{T} \phi(|f_r|) d\sigma = \lim_{r\to 1} \int_{T} \phi(|f_r|) d\sigma$$

where the last equality follows from the subharmonicity of $\phi(|f|)$.The quasi-norm $\| \ \|_{\phi}$ induces a translation invariant metric d on H_{ϕ} given by d(f,g)= $\| f-g \|_\phi$ for all $f, g \in H_\phi$. We note that H_ϕ= H(D) when ϕ is bounded. Also if ϕ is unbounded and strictly increasing ,then (H_ϕ^+ , d) is an F-space ,i .e .,a topological vector space with complete translation invariant metric (see [1] and [4]).Moreover,if $\phi(x) = x^p, 0 \le p < 1$,then H_ϕ= H^p and if $\phi(x) = \log(1 + x^p)$,then for p = 1, H_ϕ= N, $H_\phi^+ = N^+$ and for 0 < p < 1, $H_\phi^+ = N_p$ (see [2] ,[3] and [4]). In [6], N^p spaces are defined for $p \ge 1$.If we let $\phi(x) = (\log(1+x))^p, 0 < p < 1$,then we get an extension of these spaces for p>0.

In this paper we give different characterizations of the quasi-norm $\| \ \|_\phi$ similar to those in N and N^+ and a different characterization of H_ϕ (see [6]). Furthermore, we generalize these spaces to $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ where Ω is a domain ,i.e., an arbitrary open connected subset of **C** .For that purpose we use harmonic functions as in $H^p(\Omega)$,p > 0 , $N(\Omega)$ and $N^+(\Omega)$ (see [2],[3],and [7]).Also, we consider the special case Ω being finitely connected and give a factorization theorem for functions in $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ .If $H(\Omega)$ is the space of analytic functions in Ω, then we define the Hardy-Orlicz space $H_\phi(\Omega)$ to be the space of $f \in H(\Omega)$ such that $\phi(|f|)$ has a harmonic majorant in Ω ,i.e., there is a function u harmonic in Ω and $\phi(|f(z)|) \le u(z)$ *for all* $z \in \Omega$.

As in $H^p(\Omega)$ or N (Ω) for each $f \in H_\phi(\Omega)$ there is a least harmonic majorant u_f of $\phi(|f|)$,i.e., $\phi(|f(z)|) \le u_f(z)$ *for all* $z \in \Omega$ and $u_f(z) \le \upsilon(z)$ *for all* $z \in \Omega$ for any harmonic majorant υ of $\phi(|f|)$ (see [8,p.52]).

A non-negative harmonic function on Ω is called quasi-bounded if it is the pointwise increasing limit of non-negative bounded harmonic functions on Ω. We define the Hardy-Orlicz space $H_\phi^+(\Omega)$ to be the space of all $f \in H_\phi(\Omega)$ such that $\phi(|f|)$ has a quasi-bounded harmonic majorant on Ω.If $\phi(x) = x^p$, $0 < p < 1$,then $H_\phi(\Omega) = H^p(\Omega)$ and if $\phi(x) = \log(1+x)$,then $H_\phi(\Omega) = N(\Omega)$ and $H_\phi^+(\Omega) = N^+(\Omega)$ (see [2] and [8]).The special case $H_\phi^+ = H_\phi^+(D)$ is considered in [1] and [4].We note that $H^\infty(\Omega)$,the space of bounded analytic functions in Ω, is contained in $H^p(\Omega)$ for p > 0.

If z_0 is a fixed point of Ω, which we call the point of reference ,then we define the quasi-norm $\|\ \|_\phi$ on $H_\phi(\Omega)$ by

$$\| f \|_\phi = u_f(z_0)$$

for all $f \in H_\phi(\Omega)$ .The minimum principle for harmonic functions , the subadditivity of ϕ, and the sum of two harmonic functions is harmonic imply that the quasi-norm $\|\ \|_\phi$ has properties similar to those for the case Ω=D and $\phi(x) = \log(1+x)$ (see [6]).Hence,if d(f , g)= $\| f - g \|_\phi$ for all $f, g \in H_\phi(\Omega)$, then d is a translation invariant metric on $H_\phi(\Omega)$.By an easy exploitation of the analogy with $H^p(\Omega)$ and $N(\Omega)$ one can give an integrability condition on $H_\phi(\Omega)$ which is equivalent to the least harmonic majorant condition and prove that ($H_\phi(\Omega)$,d) is a complete metric space (see [8,pp.53,54]).

When ϕ is a strictly increasing unbounded modulus function we show that ($H_\phi^+(\Omega)$,d) is an F-space. This generalizes the corresponding result in [1] where Ω =D and in [2]where $\phi(x) = \log(1+x)$ .Also, as in $H^p(\Omega)$, $N(\Omega)$, and $N^+(\Omega)$ we show that $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ can be identified with closed subspaces of H_ϕ (see [2],[7],and [8]).For that purpose we need to mention the uniformization theorem for planar domains in [7,p.180].It says that if Ω has at least three boundary points ,then there exists a function φ analytic and locally 1-1 in D whose range is exactly Ω and which is invariant under a group G of linear fractional transformations of D onto itself ,i.e., $\varphi \circ g = \varphi$ for all g $\in G$ .Furthermore ,if z_0 is an arbitrary point in Ω , φ may be chosen so that φ (0)= z_0 and $\varphi'(0) > 0$ .These conditions determine φ uniquely .In other words the pair (D, φ) is the universal covering surface of Ω ,and G is the automorphic group of Ω .

2. H_ϕ and H_ϕ^+

In order to give different formulations of $\| \|_\phi$ on H_ϕ and give other characterizations of H_ϕ and H_ϕ^+ we make some definitions and quote some results in [9] .Let μ be a positive measure on a measure space X.A set $\Lambda \subseteq L^1(\mu)$ is said to be uniformly integrable if $\int_X | f | d\mu \le K < \infty$ for some constant K and $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\int_E | f | d\mu < \varepsilon$ when f $\in \Lambda$ and $\mu(E) < \delta$ .A function γ is called strongly convex if γ is convex on $(-\infty, \infty)$, $\gamma \ge 0$, γ is non-decreasing, and $\frac{\gamma(t)}{t} \to \infty \, as \, t \to \infty$.

Theorem 2.1 ([9,p. 37])

A bounded set $\Lambda \subseteq L^1(\mu)$, μ is a positive measure on a measure space X, is uniformly integrable iff $\exists$ a strongly convex γ and a constant M such that

$$\int_X \gamma(|\ f\ |)d\mu \leq M < \infty \text{ for all } f \in \Lambda$$

Theorem 2.2 ([9, p. 41])

Suppose that ψ is a subharmonic function in D, ψ is not identically $-\infty$,and C$<\infty$ is such that

$$\int_T \psi_r^+ d\sigma < C \quad (0 \leq r < 1)$$

where for z in T, $\psi_r^+(z) = 0$ if $\psi_r(z) < 0$ and $\psi_r^+(z) = \psi_r(z)$ if $\psi_r(z) \geq 0$ .Define

$$h^{(r)}(z) = \int_T P(r^{-1}z,\zeta)\psi(r\zeta)d\sigma(\zeta),\ z \in rD.$$

Then

a. $h^{(r)} \geq \psi$ in rD

b. $h^{(r)} \leq h^{(s)}$ in rD and $\int \psi_r \leq \int \psi_s$ if r < s

c. $\lim_{r \to 1} h^{(r)}$ (z) =h(z) exists for all z $\in$ D, and h is the least harmonic majorant ψ .

d. h^* exists a.e. σ, $h^* \in L^1(T)$ and $\exists$ a singular real measure υ on T such that h = P[h^* +dυ]

e. If ψ^* exists a.e. σ, then $\psi^* = h^*$ a.e. σ .

f. If $\{\psi_r^+\}$, $r \in [0,1)$, is unifomly integrable ,then $\upsilon \leq 0$,hence h $\leq$ P[h^*]

Theorem 2.3 ([10,p.85])

Let g$\in L^1(T)$, g$\geq$ 0.Then $\forall\varepsilon > 0$ $\exists\delta > 0$ such that

$$\sigma(E) < \delta \text{ ,E}\subseteq\text{T implies } \int_E g(x)dx < \varepsilon .$$

i.e.,{g} is uniformly integrable.

Now we give other characterizations of H_ϕ^+ and different ways of representing the quasi-norm on H_ϕ which motivated the definition of $H_\phi^+(\Omega)$ because [3, p.391] the quasi-bounded harmonic functions in D are exactly the Poisson integral of non-negative integrable functions on T.

Theorem 2.4

Let $f \in H^+(D) \cap H_\phi$.Then f$\in H_\phi^+$ iff $\{\phi(|f_r|)\}$,r$\in$[0,1),is uniformly integrable.

Proof : Suppose that $f \in H^+(D) \cap H_\phi$.Then

$$\| f \|_\phi = \sup_{0\le r<1}\int_T \phi(|f_r|)d\sigma = \lim_{r\to 1}\int_T \phi(|f_r|)d\sigma < \infty \tag{2.1}$$

Applying theorem 2.2 with $\psi = \phi(|f|)$ we get h = P[h^*+dυ_f] where h is the least harmonic majorant of ψ ,i.e., h = u_f .Also,

$$\text{h}(0) = u_f(0) = \lim_{r\to 1} h^{(r)}(0) = \lim_{r\to 1}\int_T \phi(|f_r|)d\sigma = \|f\|_\phi = \int_T \phi(|f|)d\sigma + \upsilon_f(\text{T}) \tag{2.2}$$

If f$\in H_\phi^+$**,** then (2.1) and (2.2) imply that υ_f(T) = 0, hence h = P[ψ^*] and

$$\phi(|f(z)|) \le \text{P}[\phi(|f(z)|)] \text{ , z} \in \text{D}$$

Therefore, for E$\subseteq$T we have

$$\int_E \phi(|\, f_r(e^{i\theta})\,|)d\theta \le \frac{1}{2\pi}\int_E\int_0^{2\pi} P_r(\theta - t)\phi(|\, f(e^{it})\,|)d\,td\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_E P_r(\theta - t)\,|)\phi(|\, f(e^{it})\,|)d\theta\,dt$$

$$=\frac{1}{2\pi}\int_{\theta-2\pi}^{\theta} P_r\,(s)(\int_E \phi(|\, f(e^{i(\theta-s)})\,|)\,d\theta\,ds$$

where s = $\theta - t$.Since for each fixed s , $0 \le \phi(|\, f(e^{i(\theta-s)})\,|) \in L^1(T)$ theorem 2.3 and translation invariance of σ imply that $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\sigma(E) < \delta \text{ ,E}\subseteq\text{T implies } \int_E \phi(|\, f_r\,|)d\sigma < \varepsilon \text{ ,r} \in [0,1) \qquad (2.3)$$

Thus,{ $\phi(|\, f_r\,|)$ },r$\in [0,1)$, is unifomly integrable .

Conversely, suppose that $f \in H^+(D) \cap H_\phi$ and, { $\phi(|\, f_r\,|)$ }, r$\in [0,1)$, is unifomly integrable .Then (2.3) holds .By Egoroff 's theorem (see [10]) there exists a set E$\subseteq$T such that

$\phi(|\, f_r\,|) \to \phi(|\, f\,|)$ as $r \to 1$ uniformly on E,

and $\sigma(T - E) < \delta$ .Hence,(2.3) gives

$$\int_T \phi(|\, f_r\,|)d\sigma < \int_E \phi(|\, f_r\,|)d\sigma + \varepsilon$$

Now uniform convergence on E implies

$$\lim_{r\to 1}\int_T \phi(|\, f_r\,|)d\sigma \le \int_E \phi(|\, f\,|)d\sigma + \varepsilon \le \int_T \phi(|\, f\,|)d\sigma + \varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary we have

$$\lim_{r\to 1}\int_T \phi(|f_r|)d\sigma \leq \int_T \phi(|f|)d\sigma \tag{2.4}$$

Also, Fatou's lemma gives

$$\int_T \phi(|f|)d\sigma \leq \lim_{r\to 1}\int_T \phi(|f_r|)d\sigma \tag{2.5}$$

Thus (2.4) and (2.5) give f$\in H_\phi^+$.

Now we give the following corollaries.

Corollary 2.5

Let $f \in H^+(D) \cap H_\phi$ .Then $\phi(|f|) \in L^1(T)$ and there exists a real singular measure υ_f such that

$$h = P[\phi(|f|) + d\upsilon_f]$$

where h is the least harmonic majorant of $\phi(|f|)$.Moreover, the following are equivilant

1. f$\in H_\phi^+$
2. h = P[$\phi(|f|)$] ,i.e., $\upsilon_f = 0$.
3. $\int_T \gamma(\phi(|f_r|))d\sigma$,r$\in[0,1)$ is bounded for some strongly convex γ .

Proof: We show that (2) implies (3) and the rest is an easy cosequence of theorems 2.1 and 2.4.So assume that (2) holds .Since $\phi(|f|) \in L^1(T)$ theorems 2.1 and 2.3 imply that there exists a strongly convex γ such that $\gamma(\phi(|f|)) \in L^1(T)$.Hence, using Jensen's inequality it follows that

$$\gamma(\phi(|f|)) \le \gamma(P[(\phi(|f|)]) \le P[\gamma(\phi(|f|))].$$

Therefore,using the properties of the Poisson kernel we have

$$\int_T \gamma(\phi(|f_r|))d\sigma \le \int_T P[\gamma(\phi(|f_r|))]d\sigma \le \int_T \gamma(\phi(|f|))d\sigma < \infty.$$

Which establishes (3).

Corollary 2.6

Let $f \in H^+(D) \cap H_\phi$ .Then f$\in H_\phi^+$ iff there exists a strongly convex γ and a harmonic function h, both non-negative such that $\gamma(\phi(|f|)) \le$ h in D.

Proof: If f$\in H_\phi^+$ **,**then corollary 2.5 implies that

$$\int_T \gamma(\phi(|f_r|))d\sigma$$

is bounded for some strongly convex γ .Since $\gamma(\phi(|f|)) = \psi$ is subharmonic theorem 2.2 gives the required h. The converse follows from the harmonicity of h and corollary 2.5 since

$$\int_T \gamma(\phi(|f_r|))d\sigma \le h(0) < \infty.$$

Finally, from above we obtain the following representations of $\| f \|_\phi$ for $f \in H^+(D) \cap H_\phi$

1. $\sup_{0 \le r < 1} \int_T \phi(|f_r|)d\sigma$
2. $\lim_{r \to 1} \int_T \phi(|f_r|)d\sigma$

3. u_f (0),where u_f is the least harmonic majorant of $\phi(|f|)$.

4. $\eta_f(T)$ where $u_f = P[d\eta_f]$ and

$$d\eta_f = \phi(|f|)d\sigma + d\upsilon_f$$

where υ_f is singular with respect to σ.

5. $\int_T \phi(|f|)d\sigma + \upsilon_f(\mathrm{T})$.

Moreover, f$\in H_\phi^+$ iff $\upsilon_f = 0$ iff u_f = P[$\phi(|f|)$] which is a quasi-bounded harmonic majorant .This motivated the definition of $H_\phi^+(\Omega)$.

3. The spaces $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$

We start by a generalization of some results in [1] from D to Ω.

Lemma 3.1

$$\bigcup_{p>1} H^p(\Omega) \subseteq H^1(\Omega) \subseteq H_\phi(\Omega) \qquad (3.1)$$

Proof: The first inclusion in (3.1) follows from $H^p(\Omega) \subseteq H^q(\Omega)$ whenever p>q>0(see [8, p.75]) .For the second inclusion in (3.1) if [x] is the greatest integer in x it is easy to show that

$$\phi(x) \le \phi(1)(1+x),\ x \ge 0 \qquad (3.2)$$

using the properties of ϕ and x $\le$ 1+ [x] for x $\ge$ 0 .

Thus (3.2) implies that if u is a harmonic majorant of |f|, then $\phi(1)$ (1+ u) is a harmonic majorant of $\phi(|f|)$.Hence, $H^1(\Omega) \subseteq H_\phi(\Omega)$.

Theorem 3.2 If $\liminf_{x\to\infty}\frac{\phi(x)}{x}=\alpha>0$,then $H^1(\Omega)=H_\phi(\Omega)$.

Proof: Suppose that $\liminf_{x\to\infty}\frac{\phi(x)}{x}=\alpha>0$ .Then there exists $x_0>0$ such that

$$x\le\frac{2}{\alpha}\phi(x),\ x\ge x_0 \tag{3.3}$$

If $f\in H_\phi(\Omega)$,then by (3.3)

$$|f(z)|\le x_0+\frac{2}{\alpha}\phi(|f(z)|)\le x_0+\frac{2}{\alpha}u(z)$$

for all $z\in\Omega$ where u is a harmonic majorant of $\phi(|f|)$ on Ω.Thus $H_\phi(\Omega)\subseteq H^1(\Omega)$ and the proof is complete by lemma 3.1.

Theorem 3.3 If $\liminf_{x\to\infty}\frac{\phi(x)}{\log x}=\alpha>0$, then $H_\phi(\Omega)\subseteq N(\Omega)$ and $H_\phi^+(\Omega)\subseteq N^+(\Omega)$.

Proof : Let $g(x)=\inf\{\frac{\phi(t)}{\log t}:t\ge x\}$ .Then $\lim_{x\to\infty}g(x)=\alpha$ implies that there exists $x_0\ge 1$ such that

$$\log x\le\frac{2}{\alpha}\phi(x),\ x\ge x_0 \tag{3.4}$$

Since $\log(1+x)\le 1+\log x$ for all $x\ge x_0$ using (3.4) we get

$$\log(1+x)\le K'+\frac{2}{\alpha}\phi(x),\ x\ge 0 \tag{3.5}$$

where K'=1+ log (1+ x_0).Hence ,for $f \in H_\phi(\Omega)$ by (3.5) we have for all z$\in \Omega$

$$\log(1+|f(z)|) \leq K' + \frac{2}{\alpha}\phi(|f(z)|) \leq K' + \frac{2}{\alpha}u(z)$$

where u is the least harmonic majorant of $\phi(|f|)$ on Ω.Thus $f \in N(\Omega)$ and hence $H_\phi(\Omega) \subseteq N(\Omega)$.The other inclusion follows from above by replacing harmonic majorant by quasi-bounded harmonic majorant.

Next we state the following result in [2,p.261] which is found to be useful for establishing certain properties of $H_\phi(\Omega)$.

Proposition 3.4 Let Ω be a domain in **C**,K a compact subset of Ω and $z_0 \in \Omega$.Then there exist positive numbers α *and* β (depending on z_0, K ,and Ω) such that

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$$

for all z$\in$K and for all u$\geq$0 with u harmonic in Ω.

Clearly proposition 3.4 implies that different points of reference induce equivalent metrics on $H_\phi(\Omega)$.Moreover, letting u = u_f in proposition 3.4 gives the following corollary as a generalization of lemma 3 in [1].

Corollary 3.5 Let K be a compact subset of Ω and $z_0 \in \Omega$.Then there exists a positive constant $\beta = \beta(z_0, K, \Omega)$ such that

$$\phi(|f(z)|) \leq \beta \| f \|_\phi, \text{ for all } f \in H_\phi(\Omega) \text{ and for all } z \in K.$$

Moreover, if ϕ is strictly increasing and unbounded, then

$$|f(z)| \leq \phi^{-1}(\beta \| f \|_\phi), \text{ for all } f \in H_\phi(\Omega) \text{ and for all } z \in K \tag{3.6}$$

where ϕ^{-1} is the inverse of ϕ.

Let $\{f_n\}$ be a sequence in $H_\phi(\Omega)$ and $f \in H_\phi(\Omega)$. We say that $f_n \to f$ in $H_\phi(\Omega)$ as $n \to \infty$ if $d(f_n, f) = \| f_n - f \|_\phi \to 0$ as $n \to \infty$. Also, we say that $f_n \overset{uc}{\to} f$ as $n \to \infty$ if $f_n \to f$ uniformly on compact subsets of Ω as $n \to \infty$.

Corollary 3.6 Let ϕ be a strictly increasing unbounded modulus function. If $f_n \to f$ in $H_\phi(\Omega)$ as $n \to \infty$, then $f_n \overset{uc}{\to} f$ as $n \to \infty$.

Proof: Use continuity of ϕ^{-1} and (3.6).

In analogy with $H^p(\Omega)$ and $N(\Omega)$ we state an integrability condition on $H_\phi(\Omega)$ which is equivalent to the least harmonic majorant condition.We omit the proof of this result as well as the proof of completeness of $H_\phi(\Omega)$ and a corollary of it because easy modification of the $H^p(\Omega)$ or $N(\Omega)$ cases gives the required results. We refer the reader to [8,pp.53,54] for definitions and proofs.

Theorem 3.7 Let ϕ be a strictly increasing unbounded modulus function and $f \in H(\Omega)$**.** Then $f \in H_\phi(\Omega)$ iff for all regular exhaustions $\{\Omega_n\}$ of Ω there exists a constant C such that

$$\int_{\partial\Omega_n} \phi(|\ f\ |)\, d\omega_{n,z} \le C < \infty \text{ , n = 1,2,3…,}$$

where $\omega_{n,z}$ is the harmonic measure on $\partial\Omega_n$,the boundary of Ω_n ,and for some point $z \in \Omega_1$.

Theorem 3.8 Let ϕ be a strictly increasing unbounded modulus function. Then($H_\phi(\Omega)$,d) is a complete metric space.Moreover,the topology in $H_\phi(\Omega)$ is stronger than that of uniform convergence on compact subsets of Ω **.**

Corollary 3.9 Let ϕ be a strictly increasing unbounded modulus function and $\{f_n\}$ is a sequence in $H_\phi(\Omega)$.If $\{\| f_n \|_\phi\}$ is bounded ,then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \overset{uc}{\to} f$ as $k \to \infty$ where $f \in H_\phi(\Omega)$.

As in $H^p(\Omega)$ and $N(\Omega)$ the uniformization theorem can be used to identify $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ with closed subspaces of H_ϕ .Let φ: D$\to \Omega$ and G be as in the uniformization theorem.Define the following subspaces of H_ϕ by

$$H_\phi / G = \{f \in H_\phi : f \circ g = f \ \ for\, all\ g \in G\}$$

and

$$H_\phi^+ / G = \{f \in H_\phi^+ : f \circ g = f \ \ for\, all\ g \in G\}$$

Also, define A:$\to H_\phi / G \ H_\phi(\Omega)$ by Af = f$\circ\, \varphi$ for all $f \in H_\phi(\Omega)$ .Then as in $H^p(\Omega)$ and $N(\Omega)$ we have the following results.

Theorem 3.10 Let ϕ be a strictly increasing unbounded modulus function. Then H_ϕ / G and H_ϕ^+ / G are closed and hence complete subspaces of H_ϕ**.**

Proof: Since by theorem 3.8, or from the definition, H_ϕ is complete it suffices to show that H_ϕ / G and H_ϕ^+ / G are closed subspaces of H_ϕ .So let $\{f_n\}$ be a sequence in H_ϕ / G such that $f_n \to f$ in H_ϕ as $n \to \infty$ and

$f \in H_\phi$.We prove that $f \in H_\phi / G$.For each $g \in G$ Harnack's inequality gives

$$\| f_n \circ g - f \circ g \|_\phi = \| (f_n - f) \circ g \|_\phi = u_{(f_n - f)\circ g}(0) \leq (u_{f_n - f} \circ g)(0) = u_{f_n - f}(g(0))$$

$$\leq \frac{1+|g(0)|}{1-|g(0)|} u_{f_n - f}(0) = \frac{1+|g(0)|}{1-|g(0)|} \| f_n - f \|_\phi$$

Thus $f_n = f_n \circ g \to f \circ g$ in H_ϕ as $n \to \infty$ .Corollary 3.6 implies that $f_n \overset{uc}{\to} f \circ g$ as $n \to \infty$ and $f_n \overset{uc}{\to} f$ as $n \to \infty$ .Thus $f \circ g = f$ for all $g \in G$ which proves that $f \in H_\phi / G$.

The completeness of H_ϕ^+ and the above argument imply that H_ϕ^+ / G is a closed subspace of H_ϕ .

The proof of the next result is similar to that in case of $H^p(\Omega)$ and $N(\Omega)$ and we omit it (see [8,p.63]).

Theorem 3.11 Let ϕ be a strictly increasing unbounded modulus function. Then A: $H_\phi(\Omega) \to H_\phi / G$ where Af = fφ $\circ$for all $f \in H_\phi(\Omega)$ is an onto isometric isomorphism.

The isometry A can be used to prove the following results.

Corollary 3.12 Let ϕ be a strictly increasing unbounded modulus function. Then

1. $H_\phi(\Omega)$ is a complete metric space and $H_\phi^+(\Omega)$ is an F-space.
2. $\bigcup_{p \geq 1} H^p(\Omega) \subseteq H_\phi^+(\Omega)\ \ H_\phi(\Omega) \subseteq$ (3.7)

Proof: The general form of Lebesgue dominated convergence theorem (see [10,p.89])and (3.2) imply that $H^1 \subseteq H_\phi^+$. Therefore,

$$\bigcup_{p\geq 1} H^p \subseteq H^1 \subseteq H_\phi^+ \subseteq H_\phi$$

and

$$\bigcup_{p\geq 1} H^p / G \subseteq H^1 / G \subseteq H_\phi^+ / G \subseteq H_\phi / G \tag{3.8}$$

where $H^p / G = \{ f \in H^p : f \circ g = f \ for\, all\ g \in G\}$,p> 0.Since [3,p.392] a non-negative harmonic function u on Ω is quasi-bounded iff $\tilde{u} = u \circ \varphi$ is quasi-bounded on D, it follows that A: $H_\phi^+(\Omega) \rightarrow H_\phi^+ / G$ is an onto isometric isomorphism.Therefore, $A^{-1}(H_\phi / G) = H_\phi(\Omega)$ is complete and $A^{-1}(H_\phi^+ / G) = H_\phi^+(\Omega)$ is an F- space .Moreover,since [8] A restricted to $H^p(\Omega)$ is an isometric isomorphism onto H^p / G , p> 0,(3.8) implies (3.7).

We note that corollary 3.12 is an improvement of lemma 3.1.

4. Ω is a multiply connected domain

We start by noting that in analogy with $H^p(\Omega)$ and $N(\Omega)$, $H_\phi(\Omega)$ is conformally invariant , i.e., if φ is a 1-1 holomorphic mapping of a domain Ω^* onto a domain Ω ,the point of reference in Ω is z_0 , and the point of reference in Ω^* is $w_0 = \varphi^{-1}(z_0)$,then $f \circ \varphi \in H_\phi(\Omega^*)$ for each $f \in H_\phi(\Omega)$ and $\| f \|_\phi = \| f \circ \varphi \|_\phi$. This is a consequence of the fact that φ carries the least harmonic majorant of $\phi(|f|)$ to the least harmonic majorant of $\phi(|f \circ \varphi|)$,i .e , $u_{f\circ\varphi} = u_f \circ \varphi$. Thus if Ω is simply connected, then $H_\phi(\Omega)$ and H_ϕ are isometrically isomorphic. Also, when Γ ,the boundary of Ω ,is a rectifiable Jordan curve each $f \in H_\phi^+(\Omega)$ has boundary values f^* see ([8,p.88]).Moreover, the following decomposition theorem for functions in $H_\phi(\Omega)$ is a generalization of those for $H^p(\Omega)$ and $N(\Omega)$ (see [2.p.236],[3,p.86],and [5,p.512]).

Theorem 4.1 Let Ω be a finitely connected domain whose boundary Γ consists of disjoint analytic simple closed curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ .Let U_k be the domain with boundary Γ_k which contains Ω ,$1 \le k \le n$ .Then for all $f \in H_\phi(\Omega)$ there exists $f_k \in H_\phi(U_k)$ such that

$$f = \sum_{k=1}^{n} f_k \quad on\, \Omega$$

Moreover, if $f \in H_\phi^+(\Omega)$ **,** then $f_k \in H_\phi^+(U_k)$ **,** $1 \le k \le n$.

Let the pair (D,φ) be the universal covering surface of Ω with $\varphi(0) = z_0$ in Ω and ω is the harmonic measure on Γ for z_0 . We point out that, as in $H^p(\Omega)$ [8,p.88], theorem 4.1 implies that each $f \in H_\phi^+(\Omega)$ has boundary values f^* and $\phi(|f^*|) \in L^1(\Gamma, \omega)$.

If a = $re^{i\theta} \in$ D and z =φ(a) ,then [8,p.50]

$$\int_\Gamma u\, d\omega_z = \frac{1}{2\pi}\int_0^{2\pi} (u \circ \varphi^*)(e^{it}) P_r(\theta - t)\, dt \ , \ u \in L^1(\Gamma, \omega) \tag{4.1}$$

where P is the Poisson kernel, φ^* is the boundary values of φ ,and ω_z is the harmonic measure on Γ for z .In particular ,if a = 0, then

$$\int_\Gamma u\,d\omega = \int_T u \circ \varphi^* d\sigma \;,\; u \in L^1(\Gamma,\omega) \tag{4.2}$$

Now we are ready to give an integrability condition for functions in $H_\phi^+(\Omega)$ which is a generalization of the special case Ω = D. Moreover, we give a formula for u_f when $f \in H_\phi^+(\Omega)$.

Theorem 4.2 Let Ω be a finitely connected domain whose boundary Γ consists of disjoint analytic simple closed curves .Then $f \in H_\phi^+(\Omega)$ iff

$$\| f \|_\phi = \int_\Gamma \phi(| f^* |)d\omega \tag{4.3}$$

Moreover, if $f \in H_\phi^+(\Omega)$, then

$$u_f(\mathrm{z}) = \int_\Gamma \phi(| f^* |)d\omega_z \,,\ \mathrm{z}\Omega \in \tag{4.4}$$

Proof: Suppose that $f \in H_\phi^+(\Omega)$ .Then by (4.2) we have

$$\|f\|_\phi = \|Af\|_\phi = \|f\circ\varphi\|_\phi = \int_T \phi(|(f\circ\varphi)^*|)d\sigma = \int_T \phi(| f^* \circ \varphi^* |)d\sigma = \int_\Gamma \phi(| f^* |)d\omega .$$

Thus (4.3) holds.

Conversely, suppose that (4.3) holds .Then

$$\| f\circ\varphi\|_\phi = \| f \|_\phi = \int_\Gamma \phi(| f^* |)d\omega = \int_T \phi(| f^* \circ \varphi^* |)d\sigma = \int_T \phi(|(f \circ \varphi)^* |)d\sigma$$

Thus $f \circ \varphi \in H_\phi^+ / G$ and $f \in H_\phi^+(\Omega)$ by the isometry A.

Next if $f \in H_\phi^+(\Omega)$, then $f \circ \varphi \in H_\phi^+ / G$ and by corollary 2.5 we have

$$u_{f\circ\phi} = P[\phi(|(f\circ\varphi)^*|)d\sigma].$$

Hence, if $\zeta = \varphi^{-1}(z)$,then (4.1) and $u_f = u_{f\circ\phi} \circ \varphi^{-1}$ imply that

$$u_f(z) = (u_{f\circ\phi} \circ \varphi^{-1})(z) = u_{f\circ\phi}(\zeta) = \int_T \phi(|(f\circ\varphi)^*(e^{it})|)P_r(\theta - t)d\sigma = \int_\Gamma \phi(|f^*|)d\omega_z$$

where $\zeta = re^{i\theta}$.

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