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Spinless particles in the field of unequal scalar–vector Yukawa potentials

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We present analytical bound state solutions of the spin-zero Klein–Gordon (KG) particles in the field of unequal mixture of scalar and vector Yukawa potentials within the framework of the approximation scheme to the centrifugal potential term for any arbitrary l-state. The approximate energy eigenvalues and unnormalized wave functions are obtained in closed forms using a simple shortcut of the Nikiforov–Uvarov (NU) method. Further, we solve the KG–Yukawa problem for its exact numerical energy eigenvalues via the amplitude phase (AP) method to test the accuracy of the present solutions found by using the NU method. Our numerical tests using energy calculations demonstrate the existence of inter-dimensional degeneracy amongst the energy states of the KG–Yukawa problem. The dependence of the energy on the dimension D is numerically discussed for spatial dimensions $D = 2$–6.

Keywords: Klein–Gordon equation, Yukawa potential, D-dimensional space, Nikiforov–Uvarov and amplitude phase methods

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1. Introduction

The Yukawa potential or static screening Coulomb (SSC) potential is often used to compute the bound-state normalizations and the energy levels of neutral atoms[1–6] which have been studied over the past years. It is known that the SSC potential yields reasonable results only for the innermost states when the atomic number Z is large. However, for the outermost and middle atomic states, it gives rather poor results. The bound-state energies of the SSC potential with $Z = 1$ have been studied in the light of the shifted large-N expansion method.[7] For example, Chakrabarti and Das presented a perturbative solution of the Riccati equation leading to an analytical superpotential for the Yukawa potential.[8] Ikhdair and Sever investigated the energy levels of neutral atoms by applying an alternative perturbative scheme in solving the Schrödinger equation for the Yukawa potential model with a modified screening parameter.[9] They also studied the bound states of the Hellmann potential, which represents the superposition of the attractive Coulomb potential $-a/r$ and the Yukawa potential $b\exp(-\delta r)/r$ of arbitrary strength b and screening parameter.[10] Some authors studied relativistic and non-relativistic equations with different potentials.[11–44]

The aim of the present work is to investigate the KG equation in an arbitrary dimension D[45] with the unequal mixture of scalar and vector Yukawa potentials:

$$V(r) = -V_0\frac{\mathrm{e}^{-ar}}{r}, \tag{1a}$$

$$S(r) = -S_0\frac{\mathrm{e}^{-ar}}{r}, \tag{1b}$$

$$S(r) = \beta V(r), \quad -1 \le \beta \le 1, \tag{1c}$$

where $V_0 = \alpha Z$, $\alpha = (137.037)^{-1}$ is the fine-structure constant, Z is the atomic number, and a is the screening parameter.[1] In addition, β is an arbitrary constant demonstrating the ratio of the scalar potential to the vector potential.[46] When $\beta = 1$, we have an equal mixture, i.e., $S(r) = V(r)$, representing the exact spin symmetric limit $\Delta(r) = S(r) - V(r) = 0$ (the potential difference is exactly zero). However, when $\beta = -1$, we have $S(r) = -V(r)$, representing the exact pseudospin (p-spin) symmetric limit $\Sigma(r) = S(r) + V(r) = 0$ (the potential sum is exactly zero). The strong singular centrifugal term is approximated within the framework of an improved approximation scheme. Further, the spinless D-dimensional KG equation with the scalar and vector Yukawa potentials is solved using the parametric generalization of the NU method[47–49] in order to obtain the approximate analytical energy eigenvalues and corresponding wave functions for any l-state. The approximated numerical energy eigenvalues in the present model are compared with exact numerical results obtained by the AP method.[50–53]

The present work is organized as follows: In Section 2,

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the generalized parametric NU and AP methods are briefly introduced. In Section 3, we give a review to the KG equation in D-dimensional space and then obtain the bound state solutions of the hyperradial KG equation with an unequal mixture of scalar and vector Yukawa potentials by using a shortcut of the NU method. The exact and approximate numerical results are also obtained by the AP and NU methods, respectively. Finally, we give our concluding remarks in Section 4.

2. Methods of analysis

In this section, we give a brief review of the analytical NU and the numerical AP methods.

2.1. Parametric NU method

This powerful mathematical tool could be used to solve the second-order differential equations. Considering the following differential equation[47–49]

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of the second degree at most, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and checking the validity of solution unnecessary we present a shortcut. We begin the method by writing the general form of the Schrödinger-like equation (2) as

$$\psi_n''(s)+\left(\frac{c_1-c_2 s}{s(1-c_3 s)}\right)\psi_n'(s)+\left(\frac{-p_2 s^2+p_1 s-p_0}{s^2(1-c_3 s)^2}\right)\psi_n(s)=0, \tag{3}$$

where the wave functions $\psi_n(s)$ satisfies

$$\psi_n(s)=\varphi(s)y_n(s). \tag{4}$$

By comparing Eq. (3) with its counterpart Eq. (2), one can obtain

$$\tilde{\tau}(s)=c_1-c_2 s,\quad \sigma(s)=s(1-c_3 s),$$
$$\tilde{\sigma}(s)=-p_2 s^2+p_1 s-p_0. \tag{5}$$

According to the NU method,[47] one can obtain the bound-state energy equation as[48,49]

$$c_2 n-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})$$
$$+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0. \tag{6}$$

In addition, we also find that

$$\rho(s)=s^{c_{10}}(1-c_3 s)^{c_{11}},\quad \varphi(s)=s^{c_{12}}(1-c_3 s)^{c_{13}},$$
$$c_{12}>0,\ c_{13}>0,$$
$$y_n(s)=\mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s),\ c_{10}>-1,\ c_{11}>-1, \tag{7a}$$

are necessary in calculating the wave functions

$$\psi_{nl}(s)=N_{nl}s^{c_{12}}(1-c_3 s)^{c_{13}}\mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s), \tag{7b}$$

where $\mathrm{P}_n^{(\mu,\nu)}(x)$, $\mu>-1$, $\nu>-1$, $x\in[-1,1]$ are Jacobi polynomials with constant parameters[30]

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),$$
$$c_6=c_5^2+p_2,\ c_7=2c_4c_5-p_1,$$
$$c_8=c_4^2+p_0,\ c_9=c_3(c_7+c_3c_8)+c_6,$$
$$c_{10}=c_1+2c_4+2\sqrt{c_8}-1>-1,$$
$$c_{11}=1-c_1-2c_4+\frac{2}{c_3}\sqrt{c_9}>-1,\ c_3\neq 0,$$
$$c_{12}=c_4+\sqrt{c_8}>0,$$
$$c_{13}=-c_4+\frac{1}{c_3}(\sqrt{c_9}-c_5)>0,\ c_3\neq 0, \tag{8}$$

with $c_{12}>0,\ c_{13}>0$ and $s\in[0,1/c_3],\ c_3\neq 0$.

In a more special case of $c_3=0$, equation (7b) becomes

$$\lim_{c_3\to 0}\mathrm{P}_n^{(c_{10},c_{11})}(1-2c_3 s)=\mathrm{L}_n^{c_{10}}(c_{11}s),$$
$$\lim_{c_3\to 0}(1-c_3 s)^{c_{13}}=\mathrm{e}^{c_{13}s},$$
$$\psi(s)=Ns^{c_{12}}\mathrm{e}^{c_{13}s}\mathrm{L}_n^{c_{10}}(c_{11}s), \tag{9}$$

where $\mathrm{L}_n^{\alpha}(x)$ are the associated Laguerre polynomials

2.2. Amplitude–phase method

The amplitude–phase (AP) method used for calculating bound states was presented by Korsch and Laurent in 1981.[50] The Schrödinger equation can be converted into an equation for the so-called amplitude function which has a formal relationship to a local wave number that determines the so-called phase function. It begins by writing the radial solution $R_{nl}(r)$ of the Schrödinger equation in the form[50–53]

$$R_{nl}(r)=u(r)\sin\varphi(r) \tag{10}$$

with an imposed relationship

$$\varphi'(r)=u(r)^{-2}. \tag{11}$$

One can derive a nonlinear second-order differential equation

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}+\left\{2[E_{nl}-V(r)]-\frac{l(l+1)}{r}\right\}u(r)=\frac{1}{u(r)^3} \tag{12}$$

for the amplitude function. Equation (12) is the Milne-type equation.[50] Here, a phase reference condition (because of an arbitrary integration constant) is required, and it is formally accomplished with

$$\varphi(r)\to 0,\quad \text{as}\quad r\to 0. \tag{13}$$

The integration of Eq. (12) goes in two steps: from the potential minimum towards $+0$, and from the potential minimum towards $+\infty$. Using initially the (first-order) semiclassical formula of the amplitude as[50]

$$u(r_{\min})\approx\left[2(E_{nl}-V(r_{\min}))-\frac{l(l+1)}{(r_{\min})^2}\right]^{-1/4},\ u'(r_{\min})=0. \tag{14}$$

As the wave function will vanish at $r\to+\infty$, the quantization condition as

$$\varphi(+\infty)=(1+n)\pi,\ n=0,1,\ldots, \tag{15}$$

where n is the (nodal) radial quantum number. Equation (15) is solved by using the Newton's iteration with respect to energy. Further details of the numerical aspects are recently given in Ref. [53].

3. The hyperradial part of the KG equation in *D*-dimensional space

In spherical coordinates, the KG equation with vector $V(r)$ and scalar $S(r)$ potentials can be written as[46] (in units of $\hbar = c = 1$)

$$\left[\Delta_D + (E_{nl} - V(r))^2 - (M + S(r))^2\right]\psi_{nlm}(r, \Omega_D) = 0 \quad (16)$$

with

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}, \quad (17)$$

where E_{nl}, Λ_D^2, $(\Omega_D)/r^2$, and Ω_D are the energy eigenvalues, the generation of the centrifugal barrier for the D-dimensional space, and the angular coordinates, respectively.[45] The eigenvalues of $\Lambda_D^2(\Omega_D)$ are given by[45]

$$\Lambda_D^2(\Omega_D)\mathrm{Y}_l^m(\Omega_D) = \frac{(D+2l-2)^2-1}{4}\mathrm{Y}_l^m(\Omega_D), \ D > 1 \quad (18)$$

where $\mathrm{Y}_l^m(\Omega_D)$ is the hyperspherical harmonic. For $D = 2$, and $D = 3$, we have

$$\Lambda_{D=2}^2(\Omega_{D=2})\mathrm{Y}_l^m(\Omega_{D=2}) = (m^2 - 1/4)\mathrm{Y}_l^m(\Omega_{D=2})$$

and a familiar form

$$\Lambda_{D=3}^2(\Omega_{D=3})\mathrm{Y}_l^m(\Omega_{D=3}) = l(l+1)\mathrm{Y}_l^m(\Omega_{D=3}),$$

respectively. Using the separation of variables by means of the wave function $\psi_{nlm}(r, \Omega_D) = r^{-(D-1)/2}R_{nl}(r)\mathrm{Y}_l^m(\Omega_D)$, equation (16) reduces to

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - (M^2 - E_{nl}^2) - 2(E_{nl}V(r) + MS(r)) + V^2(r) - S^2(r) - \frac{(D+2l-2)^2-1}{4r^2}\Big]R_{nl}(r) = 0. \quad (19)$$

Substituting the scalar and vector Yukawa potentials into Eq. (19), we obtain

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \varepsilon^2 + \frac{(V_0^2 - S_0^2)}{r^2}\mathrm{e}^{-2ar} + \frac{2(MS_0 + E_{nl}V_0)}{r}\mathrm{e}^{-ar} - \frac{(D+2l-2)^2-1}{4r^2}\Big]R_{nl}(r) = 0, \quad (20)$$

where $\varepsilon^2 = M^2 - E_{nl}^2$. We investigate the asymptotic behavior of $R_{nl}(r)$. First, equation (20) shows that when r approaches ∞, the asymptotic solution $R_0(r)$ of Eq. (20) satisfies the differential equation

$$\frac{\mathrm{d}^2R_0(r)}{\mathrm{d}r^2} - \varepsilon^2 R_0(r) = 0,$$

assuming the solution of $R_0(r) \sim a_1\,\mathrm{e}^{-\varepsilon r}$, where a_1 is a constant. The solution is an acceptable physical solution since the solution becomes finite as $r \to \infty$. Meanwhile, as $r \to 0$, the asymptotic solution $R_\infty(r)$ of Eq. (20) satisfies the differential equation,

$$\frac{\mathrm{d}^2R_\infty(r)}{\mathrm{d}r^2} - \frac{(k^2 - 1/4)}{r^2}R_\infty(r) = 0,$$

where

$$k = \frac{1}{2}\sqrt{(D+2l-2)^2 + 4(S_0^2 - V_0^2)},$$

which assumes the solution of $R_\infty(r) \sim a_2 r^{k+1/2} + a_3 r^{-(k+1/2)}$, where a_2 and a_3 are two constants. $r^{-(k+1/2)}$ is not a satisfactory solution because it becomes infinite when $r \to 0$, while the term $r^{k+1/2}$ is well-behaved. Consequently, the asymptotic behavior of $R(r)$ suggests we choose the appropriate ansatz as $R(r) = Ar^{k+1/2}\,\mathrm{e}^{-\varepsilon r}\mathrm{F}(r)$, where $\mathrm{F}(r)$ is a hypergeometric function to be found from Eq. (20) in the region $r \in (0, \infty)$.

It is obvious that equation (20) does not have an exact solution due to the singular terms $1/r$ and $1/r^2$. So we make approximations for these two terms in Eq. (20) as[36–40]

$$\frac{1}{r^2} \approx 4a^2\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})^2}, \quad (21a)$$

$$\frac{1}{r} \approx 2a\frac{\mathrm{e}^{-ar}}{(1-\mathrm{e}^{-2ar})}, \quad (21b)$$

which are valid when $ar \ll 1$.[54–58] Thus, the vector and scalar Yukawa potential in Eqs. (1a) and (1b) can be approximated as

$$V(r) = -2aV_0\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})}, \quad (22a)$$

and

$$S(r) = -2aS_0\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})}, \quad (22b)$$

respectively. To show the accuracy of our approximation, we plot the Yukawa potential of Eq. (1a) and its approximation of Eq. (22a) with parameter values $V_0 = \sqrt{2}$, $a = 0.05V_0$[59] in Fig. 1 (1 fm = 10^{-15} m).

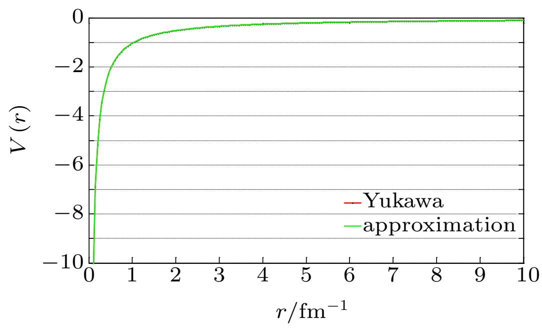

Fig. 1. (color online) The Yukawa potential (red line) and its approximation in Eq. (16) (green line).

Substituting Eq. (21) into Eq. (19), one obtains

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \varepsilon^2 + 4a^2(V_0^2 - S_0^2)\frac{\mathrm{e}^{-4ar}}{(1-\mathrm{e}^{-2ar})^2} + 4a(MS_0 + E_{n,l}V_0)\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})} - a^2(D+2l-1)(D+2l-3)\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})^2}\Big\}R_{nl}(r) = 0. \quad (23)$$

Making a suitable change of variables as $s = \mathrm{e}^{-2ar}$, and mapping $r \in (0, \infty)$ to $s \in (0, 1)$, we can recast Eq. (23) as follows:

$$\frac{\mathrm{d}^2R_{n,l}(s)}{\mathrm{d}s^2} + \frac{1-s}{s(1-s)}\frac{\mathrm{d}R_{n,l}(s)}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}\Big[-\frac{\varepsilon^2}{4a^2}(1-s)^2 + (V_0^2 - S_0^2)s^2$$

$$+\frac{(MS_0+E_{nl}V_0)}{a}s(1-s)-\frac{(D+2l-2)^2-1}{4}s\Big]R_{nl}(s)$$
$$=0. \quad (24)$$

Comparing Eq. (24) with Eq. (3), we obtain the solutions of Eq. (24)

$$c_1=1,\ p_0=\frac{\varepsilon^2}{4a^2},$$
$$c_2=1,\ p_1=\frac{2\varepsilon^2}{4a^2}+\frac{(MS_0+E_{nl}V_0)}{a}-\frac{(D+2l-2)^2-1}{4},$$
$$c_3=1,\ p_2=\frac{\varepsilon^2}{4a^2}-(V_0^2-S_0^2)+\frac{(MS_0+E_{nl}V_0)}{a}. \quad (25)$$

In addition, the values of constant coefficients c_i ($i=4,5,\ldots,13$) are obtained from Eq. (8) and displayed in Table 1. Thus, using Eq. (6), the energy eigenvalue equation can be expressed as

$$\left(2n+1+\sqrt{(D+2l-2)^2-4(V_0^2-S_0^2)}+\frac{1}{a}\sqrt{M^2-E_{nl}^2}\right)^2$$
$$=\frac{M^2-E_{nl}^2}{a^2}+\frac{4(MS_0+E_{nl}V_0)}{a}+4(S_0^2-V_0^2), \quad (26a)$$

or

$$\left(2n+1+\sqrt{(D+2l-2)^2+4(S_0^2-V_0^2)}+\frac{1}{a}\sqrt{M^2-E_{nl}^2}\right)^2$$
$$=-\left(\frac{E_{nl}}{a}-2V_0\right)^2+\left(\frac{M}{a}+2S_0\right)^2, \quad (26b)$$

as a simplification, where $-M<E_{nl}<M$. Some other approximate numerical results of energy levels for various values of dimension D and quantum numbers n and l are shown in Tables 2, 3, and 4 for $V(r)\neq S(r)$, $V(r)=S(r)$, and $V(r)=-S(r)$, respectively, where the potential parameters are chosen as $a=0.05$ fm (1 fm $=10^{-15}$ m) and $M=1.0$ fm^{-1}.[56] It can be seen from these Tables that the two inter-dimensional states are degenerate according to the relationship $(n,l,D)\to(n,l\pm1,D\mp2)$. A knowledge of $E_{nl}^{(D)}$ for $D=2$ and $D=3$ provides the necessary information to find $E_{nl}^{(D)}$ for other higher dimensions. For example, we see that $E_{2,0}^{(4)}=E_{2,1}^{(2)}$.[60] Furthermore, we noticed that the energy states become less attractive for larger n, l, and D. When $S_0=\pm V_0$ (i.e., $\beta=\pm1$), E_{21} is equal to E_{30} for any $D=2$–6. However, this degeneracy is removed for $\beta\neq\pm1$. Moreover, to show the accuracy of our approximation, we calculate the exact numerical energy eigenvalues of Eq. (20) without making any approximation to the centrifugal term by using the AP method with arbitrary n, l, and D. We obtain the percentage error

$$\left|\frac{E_{\text{approx}}-E_{\text{exact}}}{E_{\text{exact}}}\right|\times100\%$$

for a few states in Table 2 as 2.7403%, 2.4987%, and 1.8994% for $(n,l,D)=(3,1,2)$, $(3,1,3)$, and $(3,1,6)$, respectively. It is found that the exact numerical energy states obtained from Eq. (20) are in good agreement with the approximate ones obtained from Eq. (24) for a low screening regime, i.e., $ar\ll1$, since our approximation works well only for the lowest energy states.[56]

In Table 2, when $n=1$ and $l=0$, all energy states are attractive (negative) for $2\leq D\leq5$ and become less attractive with the increase of the dimension. However, when $n\geq2$ and $l\geq0$, all energy states become more repulsive (positive) with the increase of the dimension. In Table 3, for $n=1$ and $l=0$, the system is attractive for all states when $D\geq2$ and becomes less attractive with the increase of the dimension. When $n=2$ and $l=0$, the system is less attractive for all states when $2\leq D\leq5$ and repulsive for $D>5$. When $n=2$ and $l=1$, together with $n=3$ and $l=0$, the system is less attractive when $D=2,3$ and more repulsive for $D>3$ when the dimension increases. Furthermore, when $n\geq3$ and $l\geq1$, it is more repulsive for $D\geq2$ as the dimension increases. In Table 3, when $n=1$ and $l=0$, all energy states are less repulsive with the increase of the dimension. However, when $n=2$ and $l=0$, the system is less repulsive for $2\leq D\leq5$ and more attractive for $D>5$. When $n=2$ and $l=1$, together with $n=3$ and $l=0$, the system is less repulsive when $D=2,3$ and more attractive for $D>3$ for all states when the dimension increases. Finally, when $n\geq3$ and $l\geq1$, it is more attractive for $D\geq2$ as the dimension increases.

Table 1. The values of the parametric constants used to calculate the energy eigenvalues and eigenfunctions.

Constant	Analytical value
c_4	0
c_5	$-1/2$
c_6	$\frac{1}{4}+\frac{\varepsilon^2}{4a^2}-(V_0^2+S_0^2)+\frac{(MS_0+E_{n,l}V_0)}{a}$
c_7	$-\frac{2\varepsilon^2}{4a^2}-\frac{(MS_0+E_{n,l}V_0)}{a}+\frac{(D+2l-1)(D+2l-3)}{4}$
c_8	$\frac{\varepsilon^2}{4a^2}$
c_9	$-(V_0^2+S_0^2)+\frac{(D+2l-1)(D+2l-3)+1}{4}$
c_{10}	$2\sqrt{\frac{\varepsilon^2}{4a^2}}$
c_{11}	$2\sqrt{-(V_0^2+S_0^2)+\frac{(D+2l-1)(D+2l-3)+1}{4}}$
c_{12}	$\sqrt{\frac{\varepsilon^2}{4a^2}}$
c_{13}	$-\frac{1}{2}+\sqrt{-(V_0^2+S_0^2)+\frac{(D+2l-1)(D+2l-3)+1}{4}}$

Let us now calculate the corresponding eigenfunctions. We use the relationship in Eq. (7) to obtain the necessary functions

$$\rho(s)=s^{\varepsilon/a}(1-s)^{\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2}},$$
$$\phi(s)=s^{\varepsilon/2a}(1-s)^{(-1+\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})/2},$$
$$y_n(s)=\mathrm{P}_n^{(\varepsilon/a,\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})}(1-2s),$$
$$R_{nl}(s)=N_{nl}s^{\varepsilon/2a}(1-s)^{(-1+\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})/2}$$
$$\times\mathrm{P}_n^{(\varepsilon/a,\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})}(1-2s), \quad (27)$$

where N_{nl} is the normalization constant. The radial wave functions can also be rewritten in a more convenient form in terms of the potential parameters as

$$R_{nl}(r) = N_{nl}\,\mathrm{e}^{\varepsilon r}(1-\mathrm{e}^{-2ar})^{(-1+\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})/2} \times \mathrm{P}_n^{(\varepsilon/a,\sqrt{4(S_0^2-V_0^2)+(D+2l-2)^2})}(1-2\,\mathrm{e}^{-2ar}), \quad (28)$$

which behaves well at boundaries, i.e., $r=0$ and $r\to\infty$.

From Eq. (26b), we obtain the energy eigenvalues of the KG particles in two- and three-dimensional spaces, respectively, as

$$\left(2n+1+2\sqrt{\left(m-\frac{1}{2}\right)^2+(S_0^2-V_0^2)}+\frac{1}{a}\sqrt{M^2-E_{nm}^2}\right)^2 = -\left(\frac{E_{nm}}{a}-2V_0\right)^2+\left(\frac{M}{a}+2S_0\right)^2, \quad (29a)$$

$$\left(2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2+(S_0^2-V_0^2)}+\frac{1}{a}\sqrt{M^2-E_{nl}^2}\right)^2 = -\left(\frac{E_{nl}}{a}-2V_0\right)^2+\left(\frac{M}{a}+2S_0\right)^2. \quad (29b)$$

In two-dimensional space case, we inserted $l\to m-1/2$ in Eq. (29a).

Setting $V(r)\to V(r)/2E_{nl}+M\to 2\mu/\hbar^2$, and $E_{nl}-M\to E_{nl}$,[56] we can obtain the solution of the Yukawa problem in the non-relativistic limit. Here $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass where m_1 and m_2 represent the masses of the electron e and the atom Ze, respectively. In these conditions, one can obtain the non-relativistic energy eigenvalues of the Yukawa potential[40]

$$E_{nl} = -\frac{\hbar^2}{2\mu}\left[\left(n+\frac{D}{2}+l-\frac{1}{2}\right)a-\frac{\mu V_0}{\hbar^2\left(n+\frac{D}{2}+l-\frac{1}{2}\right)}\right]^2 \quad (30)$$

and the corresponding radial wave functions

$$R_{nl}(r) = N_{n,l}\,\mathrm{e}^{-\sqrt{2\mu E_{nl}}r/\hbar}(1-\mathrm{e}^{-2ar})^{l+1} \times \mathrm{P}_n^{(\sqrt{2\mu E_{nl}}/\hbar a,\,2l+1-\sqrt{2\mu E_{nl}}/\hbar a)}(1-2\,\mathrm{e}^{-2ar}). \quad (31)$$

Also, when the screening parameter a approaches zero, equation (1) reduces to a Coulomb potential. Thus, in this limit, the energy eigenvalues of Eq. (30) become the energy levels of the Coulomb interaction, i.e.,

$$E_{nl} = -\frac{\mu V_0^2}{2\hbar^2\left(n+\frac{D}{2}+l-\frac{1}{2}\right)^2}, \quad (32)$$

which is identical to the result in Refs. [45] and [58] when $D=3$.

Table 2. The energy levels of the KG particles in the field of scalar and vector Yukawa potentials for various D, n, and l values with $V_0=4.5$ and $S_0=5$.

	$E_{n,l}$/fm^{-1}									
n,l	1,0	1,0	2,0	2,0	2,1	2,1	3,0	3,0	3,1	3,1
D	NU	AP	NU	AP	NU	AP	NU	AP	NU	AP
2	−0.214913	−0.219440	0.112670	0.105122	0.174406	0.166646	0.368030	0.357228	0.415044	0.403974
3	−0.194399	−0.198948	0.129002	0.121398	0.240311	0.232333	0.380497	0.369622	0.464924	0.453590
4	−0.137038	−0.141652	0.174406	0.166646	0.317325	0.309115	0.415044	0.403974	0.522848	0.511245
5	−0.052932	−0.057646	0.240311	0.232333	0.397892	0.389478	0.464924	0.453590	0.583068	0.571249
6	0.046537	0.041712	0.317325	0.309115	0.476916	0.468366	0.522848	0.511245	0.641793	0.629861

Table 3. Bound state energy levels of the KG particle subject to scalar and vector Yukawa potentials for various D, n, and l values with $V_0=S_0=5$.

	$E_{n,l}$/fm^{-1}									
n,l	1,0	1,0	2,0	2,0	2,1	2,1	3,0	3,0	3,1	3,1
D	NU	AP	NU	AP	NU	AP	NU	AP	NU	AP
2	−0.795853	−0.784988	−0.506330	−0.493815	−0.190290	−0.194878	−0.190290	−0.194878	0.098154	0.090716
3	−0.659219	−0.660899	−0.347361	−0.351045	−0.040662	−0.046067	−0.040662	−0.046067	0.224479	0.216023
4	−0.506330	−0.508584	−0.190290	−0.194878	0.098154	0.092028	0.098154	0.092028	0.337828	0.328501
5	−0.347361	−0.350140	−0.040662	−0.046067	0.224479	0.217748	0.224479	0.217748	0.438486	0.428448
6	−0.190290	−0.193533	0.098154	0.092028	0.337828	0.330622	0.337828	0.330622	0.527179	0.516612

Table 4. Bound state energy levels of the KG particle in the field of scalar and vector Yukawa potentials for various D, n, and l values with $V_0=-5$ and $S_0=5$.

	$E_{n,l}$/fm^{-1}									
n,l	1,0	1,0	2,0	2,0	2,1	2,1	3,0	3,0	3,1	3,1
D	NU	AP	NU	AP	NU	AP	NU	AP	NU	AP
2	0.795853	0.784988	0.506330	0.493814	0.190290	0.194878	0.190290	0.194878	−0.098154	−0.090716
3	0.659219	0.660899	0.347361	0.351046	0.040661	0.046069	0.040661	0.046069	−0.224478	−0.216023
4	0.506330	0.508584	0.190290	0.194878	−0.098154	−0.092028	−0.098154	−0.092028	−0.337828	−0.328501
5	0.347361	0.350140	0.040661	0.046069	−0.224479	−0.217748	−0.224479	−0.217748	−0.438485	−0.428448
6	0.190290	0.193533	−0.098154	−0.092028	−0.337828	−0.330621	−0.337828	−0.330621	−0.527180	−0.516612

4. Concluding remarks

We have used a simple shortcut of the NU method as well as an appropriate approximation to deal with the strong and soft singular terms to obtain approximate analytical bound states of the D-dimensional KG equation for scalar and vector Yukawa potentials. Numerical tests using energy calculations show the existence of inter-dimensional degeneracy of energy states prevailing to the transformation: $(n,l,D) \rightarrow (n,l \pm 1, D \mp 2)$ as shown in Tables 2 to 4. Furthermore, it is noted that when $V_0 \neq S_0$, the weakly attractive system turns to become less attractive with the increase of the dimension D, and strongly repulsive with the increase of quantum numbers n and l. When $V_0 = S_0$, the strongly attractive system becomes less attractive with the increase of D and strongly repulsive with the increases of n and l. However, when $V_0 = -S_0$, the strongly repulsive system becomes less repulsive with the increase of D and strongly attractive with the increases of n and l. We have also calculated the exact numerical energy eigenvalues of Eq. (20) via the numerical AP method, the solution of which coincides with the approximate solution of Eq. (24). We have not encountered any cumbersome and time-consuming procedures when obtaining the numerical and analytical eigenvalues and wave functions of the problem. Our results could be widely applied to the relevant fields.

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References

[1] Yukawa H 1935 *Proc. Phys. Math. Soc. Jpn.* **17** 48
[2] McEnnan J, Kissel L and Pratt R H 1976 *Phys. Rev. A* **13** 532
[3] Mehta C H and Patil S H 1978 *Phys. Rev. A* **17** 34
[4] Dutt R and Varshni Y P 1983 *Z. Phys. A* **313** 143
[5] Dutt R and Varshni Y P 1986 *Z. Phys. D* **2** 207
[6] Lai C S and Madan M P 1984 *Z. Phys. A* **316** 131
[7] Imbo T, Pagnamenta A and Sukhatme U 1984 *Phys. Lett. A* **105** 183
[8] Chakrabarti B and Das T K 2001 *Phys. Lett. A* **285** 11
[9] Ikhdair S M and Sever R 2006 *Int. J. Mod. Phys. A* **21** 6465
[10] Ikhdair S M and Sever R 2007 *J. Mol. Struct.: Theochem* **809** 103
[11] Zhang M C 2008 *Chin. Phys. B* **17** 3214
[12] Chargui Y, Chetouani L and Trabelsi A 2010 *Chin. Phys. B* **19** 020305
[13] Zhou Y and Guo J Y 2008 *Chin. Phys. B* **17** 380
[14] Zeng X X and Li Q 2009 *Chin. Phys. B* **18** 4716
[15] Xu J J, Ni G J and Lou S Y 2011 *Chin. Phys. B* **20** 020302
[16] Li H L 2011 *Chin. Phys. B* **20** 030402
[17] Taşkın F and Koçak G 2011 *Chin. Phys. B* **20** 070302
[18] Zhou S Q 2008 *Chin. Phys. B* **17** 3812
[19] Ma Z Y and Ma S H 2012 *Chin. Phys. B* **21** 030507
[20] Chen C Y, Lu F L and You Y 2012 *Chin. Phys. B* **21** 030302
[21] Hamzavi M, Eshghi M and Ikhdair S M 2012 *J. Math. Phys.* **53** 082101
[22] Lu F L and Chen C Y 2010 *Chin. Phys. B* **19** 100309
[23] Taskin F and Koçak G 2010 *Chin. Phys. B* **19** 090314
[24] Wei G F and Chen W L 2010 *Chin. Phys. B* **19** 090308
[25] Sun G H and Dong S H 2012 *Commun. Theor. Phys.* **58** 195
[26] Ikhdair S M and Hamzavi M 2012 *Chin. Phys. B* **21** 110302
[27] Kocak M and Gonül B 2007 *Chin. Phys. Lett.* **24** 3024
[28] Su K L 1997 *Chin. Phys. Lett.* **14** 721
[29] Long C Y, Qin S J, Zhang X and Wei G F 2008 *Acta Phys. Sin.* **57** 6730 (in Chinese)
[30] Wang Z B and Zhang M C 2007 *Acta Phys. Sin.* **56** 3688 (in Chinese)
[31] Zhang M C and Wang Z B 2006 *Acta Phys. Sin.* **55** 525 (in Chinese)
[32] Zhang M C and Wang Z B 2006 *Acta Phys. Sin.* **55** 521 (in Chinese)
[33] Lu F L and Chen C Y 2004 *Acta Phys. Sin.* **53** 1652 (in Chinese)
[34] Chen G 2004 *Acta Phys. Sin.* **53** 684 (in Chinese)
[35] Qiang W C 2002 *Chin. Phys.* **11** 757
[36] Li H M 2002 *Chin. Phys.* **11** 1111
[37] Fan H Y and Sun M Z 2001 *Chin. Phys.* **10** 380
[38] Li G L, Yue R H, Shi K J and Hou B Y 2001 *Chin. Phys.* **10** 113
[39] Lu H X, Zhang Y D and Wang X Q 2000 *Chin. Phys.* **9** 325
[40] Jing H and Wu J S 2000 *Chin. Phys.* **9** 481
[41] Chen C Y 2000 *Chin. Phys.* **9** 731
[42] Liu K J, Zhang Y D, Lu H X and Pan J W 2000 *Chin. Phys.* **9** 5
[43] Qiang W C 2003 *Chin. Phys.* **12** 1054
[44] Ni Z X 1999 *Chin. Phys.* **8** 8
[45] Dong S H 2011 *Wave Equations in Higher Dimensions* (Berlin: Springer-Verlag) pp. 181–202
[46] Ikhdair S M 2011 *J. Quantum Inf. Sci.* **1** 73
[47] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Berlin: Birkhausr)
[48] Ikhdair S M 2009 *Int. J. Mod. Phys. C* **20** 25
[49] Tezcan C and Sever R 2009 *Int. J. Theor. Phys.* **48** 337
[50] Korsch H J and Laurent H 1981 *J. Phys. B: At. Mol. Phys.* **14** 4213
[51] Yano T, Ezawa Y, Wada T and Ezawa H 2003 *J. Comp. Appl. Math.* **152** 597
[52] Thylwe K E 2004 *J. Phys. A: Math. Gen.* **37** L589
[53] Thylwe K E 2009 *Eur. Phys. J. D* **54** 591
[54] Setare M R and Haidari S 2010 *Phys. Scr.* **81** 065201
[55] Aydoğdu O and Sever R 2011 *Phys. Scr.* **84** 025005
[56] Ikhdair S M 2012 *Cent. Eur. J. Phys.* **10** 361
[57] Hamzavi M, Ikhdair S M and Solaimani M 2012 *Int. J. Mod. Phys. E* **21** 1250016
[58] Hamzavi M, Movahedi M, Thylwe K E and Rajabi A A 2012 *Chin. Phys. Lett.* **29** 080302
[59] Karakoç M and Boztosun I 2006 *Int. J. Mod. Phys. E* **15** 1253
[60] Ikhdair S M and Sever R 2009 *J. Math. Chem.* **45** 1137