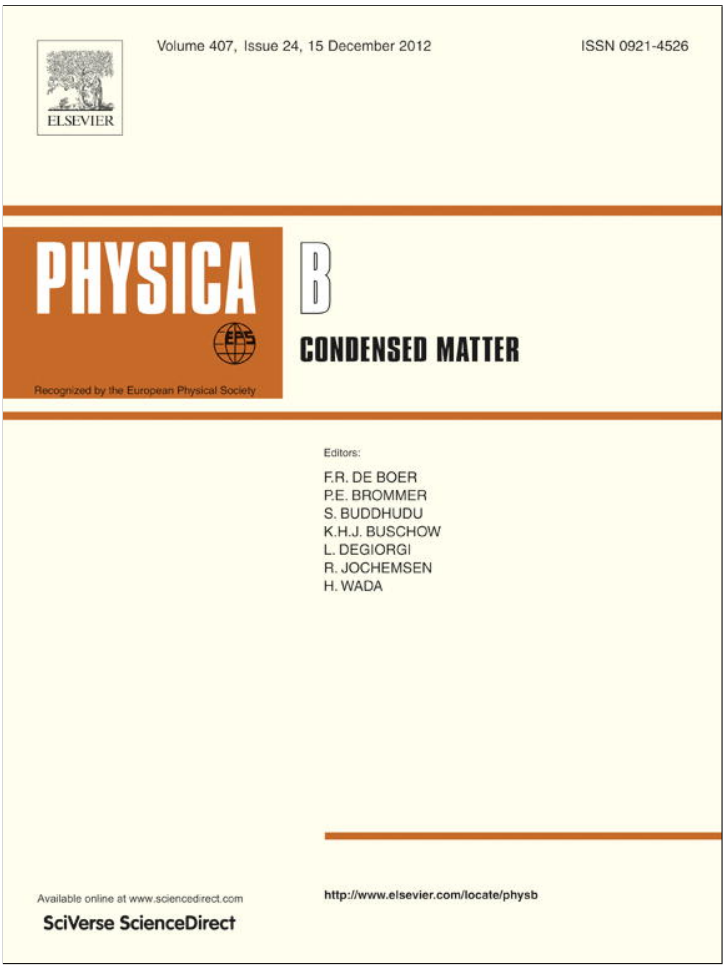

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Spectral properties of quantum dots influenced by a confining potential model

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ABSTRACT

We obtain the exact energy spectra and corresponding wave functions of the spherical quantum dots for any (n,l) state in the presence of a combination of pseudo-harmonic, Coulomb and linear confining potential terms within the exact analytical iteration method (EAIM). The interaction potential model under consideration is labeled as the Cornell modified-plus-harmonic (CMpH) type which is a correction form to the harmonic, Coulomb and linear confining potential terms.

1. Introduction

In the last decades, low-dimensional quantum systems have been the focus of extensive theoretical investigations as the subject of quantum dots (QDs). Many efforts have recently been done into understanding their electronic, optical and magnetic properties. The application of magnetic field is equivalent to introducing an additional confining potential which modifies the transport and optical properties of conduction-band electrons in QDs. In addition, introducing electrical field gives rise to electron redistribution that makes change to the energy of quantum states which experimentally control and modulate the intensity of optoelectronic devices [1]. The problem of the inverse-power potential, $1/r^n$, has been used on the level of both classical and quantum mechanics. Some series of inverse power potentials are applicable to the interatomic interaction in molecular physics [2,3]. The interaction in one-electron atoms, hadronic and Rydberg atoms takes into account inverse-power potentials [4]. Indeed, it has also been used for the magnetic interaction between spin-1/2 particles with one or more deep wells [5]. The analytical exact solutions of this class of inverse-power potentials $V(r)=Ar^{-4}+Br^{-3}+Cr^{-2}+Dr^{-1}$, $A>0$, were presented by Barut *et al* [6] and Özçelik *et al* [7] by making an available ansatz for the eigenfunctions. The Laurent series solutions of the Schrödinger equation for power and inverse-power potentials with two coupling constants $V(r)=Ar^2+Br^{-4}$ and three coupling constants $V(r)=Ar^2+Br^{-4}+Cr^{-6}$ are obtained [8,9].

The analytical exact iteration method (AEIM) which demands making a trial ansatz for the wave function [7] is general enough to be applicable to a large number of power and inverse-power potentials [10]. Recently, this method is applied to a class of power and inverse-power confining potentials of three coupling constants and containing harmonic oscillator, linear and Coulomb confining terms [11]. This kind of Cornell-plus-Harmonic (CpH) confining potential of the form $V(r)=ar^2+br-c/r$ is mostly used to study individual spherical quantum dots in semiconductors [12–14]. So far, such potentials containing quadratic, linear and Coulomb terms have been extensively studied [15,16].

The present work considers the following confining interaction potential consisting of a sum of pseudo-harmonic, linear and Columbic potential terms:

$$V(r)=V_{\mathrm{H}}(r)+V_{\text{C-mod}}(r)=ar^2+br-\frac{c}{r}-\frac{d}{r^2},a>0 \tag{1}$$

where a, b, c and d are arbitrary constant parameters to be determined latter. We will refer to this potential model in (1) as a Cornell-modified plus harmonic (CMpH) potential, since the functional form has been improved by the additional $-d/r^2$ piece; besides the contribution from the additional term also alters the value of b and c [17]. Obviously, the harmonic or power-law term has not been considered by the authors of Ref. [18] as the results are expected to be similar. For the sake of comparison, we have plotted the CpH and CMpH potentials in Fig. 1 for the values of potential parameters: $a=1.0$ eV fm^{-2}, $b=0.217$ eV fm^{-1},

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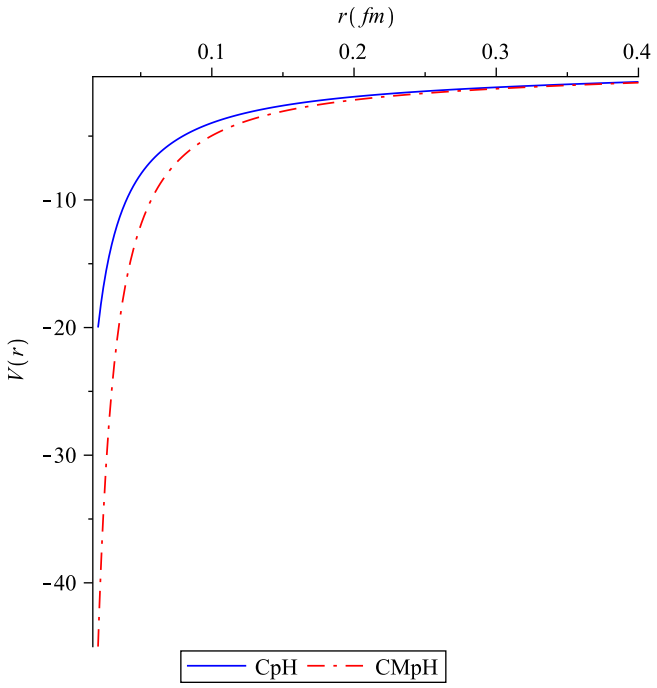

Fig. 1. A comparison between the CpH and CMpH potentials [see Eq. (1)] with the selected values of parameters: $a=1.0$ eV fm^{-2},$b=0.217$ eV fm^{-1}, $c=0.400$ eV fm and $d=0.010$ eV fm^2.

$c=0.400$ eV fm and $d=0.010$ eV fm^2. It is seen that the CMpH potential is more singular than CpH potential when $r\to 0$.

In this work, we will apply the AEIM used in [7,11] to obtain the exact energy eigenvalues and their corresponding radial wave functions of the radial Schrödinger equation (RSE) with the CMpH potential for any arbitrary (n,l) state. Very recently, we have studied the exact analytical bound state energy eigenvalues and normalized wave functions of the non-relativistic and spinless relativistic equations with pseudo-harmonic interaction under the effect of external uniform magnetic field and Aharonov–Bohm (AB) flux field in the framework of the Nikiforov–Uvarov (NU) method [19–21].

The paper is structured as follows: In Section 2, we obtain the exact energy eigenvalues and corresponding wave functions of the RSE in three-dimensions (3D) for the confining CMpH potential model by proposing a suitable form for the wave function. In Section 3, we apply our results to an electron in spherical quantum dot of InGaAs semiconductor. The relevant conclusion is given in Section 4.

2. Exact solution of RSE with a confining potential model

The three-dimensional (3D) Schrödinger equation takes the form [22]

$$\left[-\frac{\hbar^2}{2m}\Delta+V(r)\right]\psi(r,\theta,\varphi)=E_{nl}\psi(r,\theta,\varphi) \tag{2a}$$

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{L^2(\theta,\varphi)}{\hbar^2 r^2} \tag{2b}$$

where m is the isotropic effective mass and E_{nl} is the total binding energy of the particle. The complete wave function, $\psi(r,\theta,\varphi)$in (2a) can be written as:

$$\psi(r,\theta,\varphi)=\sum_{n,l}N_l\psi_{nl}(r)Y_{lm}(\theta,\varphi) \tag{3}$$

with $Y_{lm}(\theta,\varphi)$ is the spherical harmonic part of the wave function satisfying

$$L^2(\theta,\varphi)Y_{lm}(\theta,\varphi)=\hbar^2 l(l+1)Y_{lm}(\theta,\varphi) \tag{4}$$

and the radial part of the wave function, $\psi_{nl}(r)$, is the solution of the equation

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+\frac{2m}{\hbar^2}(E_{nl}-V(r))\right]\psi_{nl}(r)=0 \tag{5}$$

where r stands for the relative radial coordinates. The radial wave function $\psi_{nl}(r)$ is well-behaved at the boundaries (the finiteness of the physical solution demands that $\psi_{nl}(0)=\psi_{nl}(r\to\infty)\simeq 0$). Now, employing the transformation

$$\psi_{nl}(r)=\frac{1}{r}U_{nl}(r) \tag{6}$$

reduces Eq. (5) to the simple form

$$U''_{nl}(r)+\left[\varepsilon_{nl}-a_1r^2-b_1r+\frac{c_1}{r}+\frac{d_1-l(l+1)}{r^2}\right]U_{nl}(r)=0 \tag{7}$$

where $U_{nl}(r)$ is the reduced radial wave function and we used the simplifications:

$$\varepsilon_{nl}=\frac{2m}{\hbar^2}E_{nl},\ a_1=\frac{2m}{\hbar^2}a,\ b_1=\frac{2m}{\hbar^2}b,\ c_1=\frac{2m}{\hbar^2}c,\ d_1=\frac{2m}{\hbar^2}d \tag{8}$$

The analytical exact iteration method (AEIM) requires making the following ansatze for the wave function [9],

$$U_{nl}(r)=f_n(r)\exp[g_l(r)] \tag{9}$$

with

$$f_n(r)=\begin{cases}1, & n=0,\\ \prod_{i=1}^{n}\left(r-\alpha_i^{(n)}\right) & n=1,2,\cdots,\end{cases} \tag{10a}$$

$$g_l(r)=-\frac{1}{2}\alpha r^2-\beta r+\delta\ln r,\quad \alpha>0,\beta>0 \tag{10b}$$

It is clear that $f_n(r)$ are equivalent to the Laguerre polynomials [23]. Substituting Eq. (9) into Eq. (5) we obtain

$$U''_{nl}(r)=\left[g''_l(r)+g'^2_l(r)+\frac{f''_n(r)+2f'_n(r)g'_l(r)}{f_n(r)}\right]U_{nl}(r) \tag{11}$$

and comparing Eq. (11) with its counterpart Eq. (7) yields

$$a_1r^2+b_1r-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}-\varepsilon_{nl}=g''_l(r)+g'^2_l(r)+\frac{f''_n(r)+2f'_n(r)g'_l(r)}{f_n(r)} \tag{12}$$

The simplest case when $n=0$ requires taking $f_0(r)$and $g_l(r)$ given in Eq. (10b) to solve Eq. (12)

$$a_1r^2+b_1r-\varepsilon_{0l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r$$
$$-\alpha[1+2(\delta+0)]+\beta^2-\frac{2\beta(\delta+0)}{r}+\frac{\delta(\delta-1)}{r^2} \tag{13}$$

Further, comparing the corresponding powers of r on both sides of Eq. (13) we find the following energy formula and the restrictions on the potential parameters as:

$$\alpha=\sqrt{a_1}, \tag{14a}$$

$$\beta=\frac{b_1}{2\sqrt{a_1}},a_1>0, \tag{14b}$$

$$c_1=2\beta(\delta+0), \tag{14c}$$

$$\delta=\frac{1}{2}(1\pm l'),\ l'=\sqrt{(2l+1)^2-\frac{8md}{\hbar^2}}, \tag{14d}$$

$$\varepsilon_{0l}=\alpha\left[1+2(\delta+0)\right]-\beta^2. \tag{14e}$$

Actually, to have well-behaved solutions of the radial wave function at the boundaries; namely, the origin and the infinity, we need to take δ from Eq. (14d) as

$$\delta=\frac{1}{2}\left(1+\sqrt{(2l+1)^2-\frac{8md}{\hbar^2}}\right). \tag{15}$$

From Eqs. 14a–e together with Eqs. (8) and (15), the lowest (ground) state energy is given as follows:

$$E_{0l}=\sqrt{\frac{\hbar^2a}{2m}}(2+l')-\frac{2mc^2}{\hbar^2(1+l')^2} \tag{16}$$

where the parameter c of potential (1) should satisfy the following restriction:

$$c=\frac{b}{2}\sqrt{\frac{\hbar^2}{2ma}}(1+l') \tag{17}$$

In substituting the values of α, β and δ from Eqs. (14a), (14b) and (15), respectively, together with the parameters given in Eq. (8) into Eq. (9) and (10), we finally obtain the following ground state wave function:

$$\psi_{0l}(r)=N_{0l}r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\frac{2mc}{\hbar^2(1+l')}r\right), \tag{18a}$$

$$N_{0l}=\frac{1}{\sqrt{\Gamma(l')D_{-l'}\left(\frac{4mc}{\hbar^2(1+l')}\sqrt{\frac{\hbar}{2\sqrt{2ma}}}\right)}}\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{l'/4}\exp\left(-\frac{1}{2}\sqrt{\frac{2m}{\hbar^2a}}\frac{mc^2}{\hbar^2(1+l')^2}\right), \tag{18b}$$

where $D_\nu(z)$ are the parabolic cylinder functions [24]. Note that the above solutions are well-behaved at the boundaries, i.e., the regular solution near the origin could be $U_{nl}(r\to0)\to r^{(1+l')/2}$ and asymptotically at infinity as $U_{nl}(r\to\infty)\to\exp(-\alpha r^2-\beta r)\to0$. When $b=0$ ($c=0$), the problem turns to become the commonly known pseudoharmonic oscillator (PHO) interaction ($a=m\omega^2/2$) and consequently $\alpha=m\omega$, $\beta=b/\omega$ and $c=b\delta/(m\omega)$ yielding the ground energy state $E_{0l}=(2+l')\frac{\hbar\omega}{2}-\frac{2mc^2}{\hbar^2(1+l')^2}$ and the corresponding wave function $\psi_{0l}(r)=N_lr^{(-1+l')/2}\exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}r^2-\frac{2mc}{\hbar^2(1+l')}r\right)$, where $N_{0l}=\frac{1}{\sqrt{\Gamma(l')D_{-l'}\left(\frac{4m\omega}{\hbar^2(1+l')}\sqrt{\frac{\hbar}{2m\omega}}\right)}}\left(\frac{2m\omega}{\hbar}\right)^{l'/4}\exp\left(-\frac{mc^2}{4\hbar^3\omega(1+l')^2}\right)$.

The formula (17) is a relationship between parameters of the potential a, b, c and d. Therefore, the solutions (16) and (18) are valid for the potential parameters satisfying the restriction (17). Moreover, the relation between the potential parameters (17) depends on the orbital quantum number l which means that the potential has to be different for various quantum numbers. In applying the AEIM, the obtained solution for any potential is found to be subjected to certain restrictions on potential parameters as can be traced in other works (see, for example, [7–9,11]).

Secondly, for the first node ($n=1$), using $f_1(r)=\left(r-\alpha_1^{(1)}\right)$ and $g_l(r)$from Eq. (10b) to solve Eq. (12),

$$a_1r^2+b_1r-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}-\varepsilon_{1l}=-\alpha\left[1+2(\delta+1)\right]+\alpha^2r^2+\beta^2+\frac{\delta(\delta-1)}{r^2}+2\alpha\beta r-\frac{2\beta\delta}{r}-\frac{2\left[\beta+\alpha\alpha_1^{(1)}\right]}{r-\alpha_1^{(1)}}+\frac{2\delta}{r(r-\alpha_1^{(1)})}, \tag{19a}$$

and multiplying both sides of Eq. (19a) by $(r-\alpha_1^{(1)})$, the relationship between the potential parameters and coefficients α, β, δ and $\alpha_1^{(1)}$are

$$a_1r^2+b_1r-\varepsilon_{1l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r-\alpha\left[1+2(\delta+1)\right]+\beta^2-\frac{2\left[\beta(\delta+1)+\alpha\alpha_1^{(1)}\right]}{r}+\frac{\delta(\delta-1)}{r^2}. \tag{19b}$$

Therefore, the relations between the potential parameters and the coefficients α, β, δ and $\alpha_1^{(1)}$ are calculated as

$$\alpha=\sqrt{a_1},\beta=\frac{b_1}{2\sqrt{a_1}},\quad a_1>0,\delta=\frac{1}{2}(1+l'),\varepsilon_{1l}=\alpha\left[1+2(\delta+1)\right]-\beta^2.$$

$$c_1-2\beta(\delta+1)=2\alpha\alpha_1^{(1)},\ (c_1-2\beta\delta)\alpha_1^{(1)}=2\delta,\ d_1=l(l+1)-\delta(\delta-1), \tag{20}$$

where c_1 and $\alpha_1^{(1)}$are found from the constraint relations,

$$c=\frac{b}{2\sqrt{\frac{2ma}{\hbar^2}}}(2+l')+\sqrt{\frac{\hbar^2b^2}{8ma}+\frac{\hbar^2}{m}\sqrt{\frac{\hbar^2a}{2m}}(1+l')}, \tag{21a}$$

$$\alpha\alpha_1^{(1)2}+\beta\alpha_1^{(1)}-\delta=0\to\alpha_1^{(1)}=-\frac{b}{4a}+\sqrt{\left(\frac{b}{4a}\right)^2+\frac{\hbar(1+l')}{2\sqrt{2ma}}}. \tag{21b}$$

Hence, the energy eigenvalues are

$$E_{1l}=\sqrt{\frac{\hbar^2a}{2m}}(4+l')-\frac{b^2}{4a},$$

$$b=2\sqrt{\frac{2ma}{\hbar^2}}\frac{(2+l')c}{(1+l')(3+l')}\left[1+\sqrt{1+\left(\frac{\hbar^2}{mc^2}\sqrt{\frac{\hbar^2a}{2m}}(1+l')-1\right)\frac{(1+l')(3+l')}{(2+l')^2}}\right], \tag{22}$$

and the corresponding wave functions,

$$\psi_{1l}(r)=N_{1l}\left(r-\alpha_1^{(1)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\sqrt{\frac{m}{2\hbar^2a}}br\right), \tag{23}$$

with the normalization constant:

$$N_{1l}=\frac{\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{l'/4}\exp\left(-\frac{1}{16}\sqrt{\frac{2m}{a^2}}\hbar^3b^2\right)}{\sqrt{\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{-1}\Gamma(l'+2)S_1+\alpha_1^{(1)^2}\Gamma(l')S_2-2\left(2\sqrt{\frac{2ma}{\hbar^2}}\right)^{-1/2}\alpha_1^{(1)}\Gamma(l'+1)S_3}},$$

where

$$S_1=D_{-(l'+2)}\left(\sqrt{\frac{\hbar}{2a}}\sqrt{\frac{2m}{a}}b\hbar\right),\ S_2=D_{-l'}\left(\sqrt{\frac{\hbar}{2a}}\sqrt{\frac{2m}{a}}b\hbar\right),$$

$$S_3=D_{-(l'+1)}\left(\sqrt{\frac{\hbar}{2a}}\sqrt{\frac{2m}{a}}b\hbar\right),$$

and $\alpha_1^{(1)}$is given in Eq. (21b). If there is a PHO interaction, the energy states read

$$E_{1l}=(4+l')\frac{\hbar\omega}{2}-\frac{b^2}{2m\omega^2}, \tag{24}$$

and the corresponding wave functions,

$$\psi_{1l}(r)=N_{1l}(r-\alpha_1^{(1)})r^{(-1+l')/2}\exp\left(-\frac{1}{2}m\omega r^2-\frac{b}{\omega}r\right), \tag{25}$$

with

$$N_{1l}=\frac{\left(\frac{2m\omega}{\hbar}\right)^{l'/4}\exp\left(-\frac{1}{4}\frac{\hbar^3b^2}{m\omega^3}\right)}{\sqrt{\left(\frac{2m\omega}{\hbar}\right)^{-1}\Gamma(l'+2)S_1+\alpha_1^{(1)^2}\Gamma(l')S_2-2\left(\frac{2m\omega}{\hbar}\right)^{-1/2}\alpha_1^{(1)}\Gamma(l'+1)S_3}},$$

$$S_1=D_{-(l'+2)}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),\ S_2=D_{-l'}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),$$

$$S_3=D_{-(l'+1)}\left(\sqrt{\frac{2\hbar}{m\omega}}\frac{b\hbar}{\omega}\right),$$

and

$$b=\frac{2m\omega}{\hbar}\frac{(2+l')c}{(1+l')(3+l')}\left[1+\sqrt{1+\left(\frac{\hbar^3\omega}{2mc^2}(1+l')-1\right)\frac{(1+l')(3+l')}{(2+l')^2}}\right],$$

where $\alpha_1^{(1)}=(l'+1)/(2m\omega)$.

Following the analytical iteration procedures for the second node ($n=2$) with $f_2(r)=\left(r-\alpha_1^{(2)}\right)\left(r-\alpha_2^{(2)}\right)$ and $g_l(r)$ as defined in Eq. (10b), we obtain

$$a_1r^2+b_1r-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}-\varepsilon_{1l}=$$
$$-\alpha+\alpha^2r^2+\beta^2+\frac{\delta(\delta-1)}{r^2}+2\alpha\beta r-2\alpha\delta$$
$$-\frac{2\beta\delta}{r}-\frac{2+2(-\alpha r-\beta+\delta/r)(2r-\alpha_1^{(2)}-\alpha_2^{(2)})}{(r-\alpha_1^{(2)})(r-\alpha_2^{(2)})},\tag{26a}$$

and by multiplying both sides of Eq. (26a) in $(r-\alpha_1^{(2)})(r-\alpha_2^{(2)})$, the relationship between the potential parameters and coefficients α, β, δ, $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$are

$$a_1r^2+b_1r-\varepsilon_{2l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r$$
$$-\alpha[1+2(\delta+2)]+\beta^2-\frac{2\left[\beta(\delta+2)+\alpha\sum_{i=1}^{2}\alpha_i^{(2)}\right]}{r}+\frac{\delta(\delta-1)}{r^2},\tag{26b}$$

Therefore, the relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(2)}$and $\alpha_2^{(2)}$ are

$$\alpha=\sqrt{a_1},\cdot\beta=\frac{b_1}{2\sqrt{a_1}},\cdot\delta=\frac{1}{2}(1+l'),\cdot\varepsilon_{2l}=\alpha[1+2(\delta+2)]-\beta^2,$$

$$c_1-2\beta(\delta+2)=2\alpha\sum_{i=1}^{2}\alpha_i^{(2)},\ (c_1-2\beta\delta)\sum_{i<j}^{2}\alpha_i^{(2)}\alpha_j^{(2)}=2\delta\sum_{i=1}^{2}\alpha_i^{(2)},$$

$$[c_1-2\beta(\delta+1)]\sum_{i=1}^{2}\alpha_i^{(2)}=4\alpha\sum_{i<j}^{2}\alpha_i^{(2)}\alpha_j^{(2)}+2(2\delta+1),\tag{27}$$

The coefficients $\alpha_1^{(2)}$and $\alpha_2^{(2)}$are found from the constraint relations,

$$\alpha\sum_{i=1}^{2}\alpha_i^{(2)2}+\beta\sum_{i=1}^{2}\alpha_i^{(2)}-(2\delta+1)=0,\tag{28a}$$

$$\delta\sum_{i=1}^{2}\alpha_i^{(2)2}-\left(\beta\sum_{i=1}^{2}\alpha_i^{(2)}+1\right)\sum_{j<k}^{2}\alpha_i^{(2)}\alpha_k^{(2)}-2\alpha\sum_{j<k}^{2}\alpha_i^{(2)^2}\alpha_k^{(2)2}=0.\tag{28b}$$

Hence, the energy eigenvalues,

$$E_{2l}=\sqrt{\frac{\hbar^2a}{2m}}(6+l')-\frac{b^2}{4a},\tag{29}$$

and the associated wave functions,

$$\psi_{2l}(r)=N_l\prod_{i=1}^{2}\left(r-\alpha_i^{(2)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\sqrt{\frac{m}{2\hbar^2a}}br\right),\tag{30}$$

where $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$should satisfy the restriction relations (28a) and (28b).

We apply the present method for the third node ($n=3$) by taking $f(r)=\left(r-\alpha_1^{(3)}\right)\left(r-\alpha_2^{(3)}\right)\left(r-\alpha_3^{(3)}\right)$and $g_l(r)$ as defined in Eq. (10b) to obtain

$$a_1r^2+b_1r-\varepsilon_{3l}-\frac{c_1}{r}+\frac{l(l+1)-d_1}{r^2}=\alpha^2r^2+2\alpha\beta r-\alpha[1+2(\delta+3)]$$
$$+\beta^2-\frac{2\left[\beta(\delta+3)+\alpha\sum_{i=1}^{3}\alpha_i^{(3)}\right]}{r}+\frac{\delta(\delta-1)}{r^2},\tag{31}$$

The relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(3)}$, $\alpha_2^{(3)}$and $\alpha_3^{(2)}$ are

$$\alpha=\sqrt{a_1},\ \beta=\frac{b_1}{2\sqrt{a_1}},\ \delta=\frac{1}{2}(1+l'),\ \varepsilon_{3l}=\alpha[1+2(\delta+3)]-\beta^2,$$

$$c_1-2\beta(\delta+3)=2\alpha\sum_{i=1}^{3}\alpha_i^{(3)},$$
$$(c_1-2\beta\delta)\sum_{i<j<k}^{3}\alpha_i^{(3)}\alpha_j^{(3)}\alpha_k^{(3)}=2\delta\sum_{i<j}^{3}\alpha_i^{(3)}\alpha_j^{(3)},$$
$$[c_1-2\beta(\delta+2)]\sum_{i=1}^{3}\alpha_i^{(3)}=4\alpha\sum_{i<j}^{3}\alpha_i^{(3)}\alpha_j^{(3)}+3(2\delta+2).\tag{32}$$

The coefficients $\alpha_1^{(3)}$, $\alpha_2^{(3)}$ and $\alpha_3^{(2)}$ are found from the constraint relation,

$$\alpha\sum_{i=1}^{3}\alpha_i^{(3)^2}+\beta\sum_{i=1}^{3}\alpha_i^{(3)}-3(\delta+1)=0,\tag{33}$$

The energy eigenvalue formula is

$$E_{3l}=\sqrt{\frac{\hbar^2a}{2m}}(8+l')-\frac{b^2}{4a},\tag{34}$$

and the wave functions are:

$$\psi_{3l}(r)=N_l\prod_{i=1}^{3}\left(r-\alpha_i^{(3)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\sqrt{\frac{m}{2\hbar^2a}}br\right)\tag{35}$$

We can repeat this iteration procedure several times to write the exact energy formula for the CMpH potential with any arbitrary n state as

$$E_{nl}=\sqrt{\frac{\hbar^2a}{2m}}(2+2n+l')-\frac{b^2}{4a},\tag{36}$$

and the corresponding wave functions for any n state is

$$\psi_{nl}(r)=N_l\prod_{i=1}^{n}\left(r-\alpha_i^{(n)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2-\sqrt{\frac{m}{2\hbar^2a}}br\right).\tag{37}$$

The relations between the potential parameters and the coefficients α, β, δ, $\alpha_1^{(n)}$, $\alpha_2^{(n)}$,$\cdots$,$\alpha_3^{(n)}$are

$$\alpha=\sqrt{a_1},\ \beta=\frac{b_1}{2\sqrt{a_1}},\ \delta=\frac{1}{2}(1+l'),\ \varepsilon_{nl}=\alpha[1+2(\delta+n)]-\beta^2,$$

$$c_1-2\beta(\delta+n)=0,(n=0)$$
$$c_1-2\beta(\delta+n)=2\alpha\sum_{i=1}^{n}\alpha_i^{(n)},n=1,2,3,\cdots,$$

$$[c_1-2\beta(\delta+n-1)]\sum_{i=1}^{n}\alpha_i^{(n)}=n(2\delta+n-1),(n=1),$$

$$[c_1-2\beta(\delta+n-1)]\sum_{i=1}^{n}\alpha_i^{(n)}=4\alpha\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)}+n(2\delta+n-1),\ n=2,3,4,\cdots,$$

$$[c_1-2\beta(\delta+n-2)]\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)}=(n-1)(2\delta+n-2)+\sum_{i=1}^{n}\alpha_i^{(n)},\ (n=2),$$

$$(c_1-2\beta\delta)\sum_{i<j<k}^{n}\alpha_i^{(n)}\alpha_j^{(n)}\alpha_k^{(n)}=2\delta\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)},(n=3),$$

$$[c_1-2\beta(\delta+n-2)]\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)}=(n-1)(2\delta+n-2)\sum_{i<j}^{n}\alpha_i^{(n)}\alpha_j^{(n)}+4\alpha\sum_{i<j<k}^{n}\alpha_i^{(n)}\alpha_j^{(n)}\alpha_k^{(n)},\ n=3,4,5,\cdots, \tag{38}$$

and so on.

3. Some special cases

Now, we consider a special case of potential (1) and an application to our results. For example, when $b=0$ which provides that $c=0$, then the potential turns to become the PHO, i.e., $V_{\rm PH}(r)=(1/2)m\omega^2r^2-d/r^2$. Thereby, the energy difference between the excited and the ground states is:

$$\Delta E=E_{1l}-E_{0l}=(4+l')\frac{\hbar\omega}{2}-(2+l')\frac{\hbar\omega}{2}=\hbar\omega, \tag{39}$$

which can be used to calculate the values of the potential parameters for the desired system.

We now apply the present results to describe a realistic physical system called indium gallium arsenide (InGaAs) quantum dot, i.e., a piece of this material of a spherical form which is considered as a semiconductor composed of indium, gallium and arsenic [11]. It is used in high-power and high-frequency (say, $\omega\sim10^{15}$ Hz) electronics because of its superior electron velocity with respect to the more common semiconductors silicon and gallium arsenide. InGaAs band gap also makes it the detector material of choice in optical fiber communication at 1300 nm and 1550 nm. The gallium indium arsenide (GaInAs) is an alternative name for InGaAs. In Fig. 2, we plot the ground state electron energy

$$E_{0l}=\left(2+\sqrt{(2l+1)^2-\frac{8md}{\hbar^2}}\right)\frac{\hbar\omega}{2}-\frac{2mc^2}{\hbar^2}\left(1+\sqrt{(2l+1)^2-\frac{8md}{\hbar^2}}\right)^{-2}, \tag{40}$$

versus ω in the interval $2\times10^{14}\le\omega\le10\times10^{14}$ Hz taking the value of $c=0.001$ eV fm and $d=0$ eV fm^2 for the cases $l=0$ and $l=1$, respectively (harmonic, Coulomb and linear combination terms, i.e., CpH potential). Also, we take instead the value of the parameter $d=0.5$ eV fm^2 (pseudo-harmonic, Coulomb and linear combination terms, i.e., CMpH potential). The effective mass of electron in the InGaAs semiconductor has been chosen as $m=0.05m_e$ and $\hbar=6.5821\times10^{-16}$ eV s. It is seen from Fig. 2 how the increase in the value of ω leads to an increase in the energy of electron. The flexibility in the adjustment of the parameter d allows one to fit the spectrum of the desired model properly (cf. Fig. 2). The energy spectrum fan up/down according to the choice of potential parameter d. The parameter d should satisfy the condition $d\le(2l+1)^2\hbar^2/(8m)$. In Fig. 3 we plot the

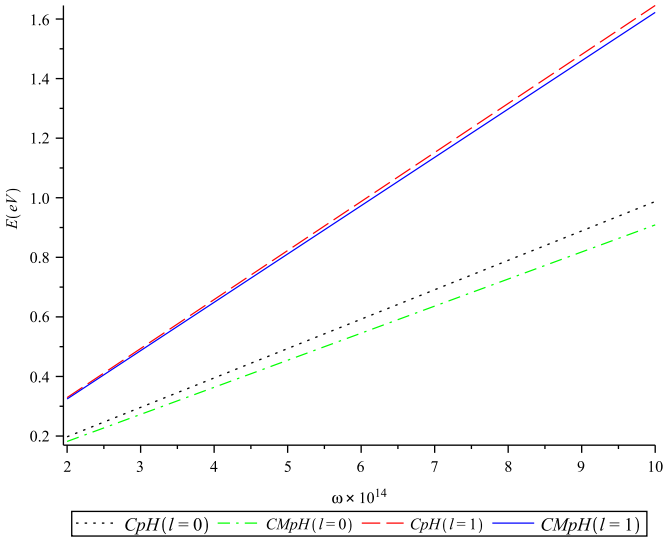

Fig. 2. Comparison of the ground state electron energy in InGaAs semiconductor versus ω in the field of the CpH and CMpH potential with $c=0.001$ eV nm and $d=0.5$ eV nm^2 for the cases $l=0$and $l=1$, respectively.

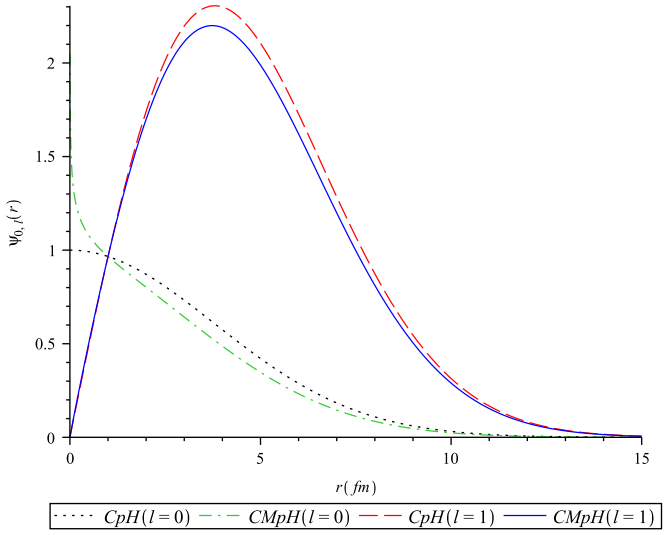

Fig. 3. Behaviour of the ground state wave function $\psi_{n=0,l=0}(r)$ and $\psi_{n=0,l=1}(r)$ of the CpH and CMpH potentials with the values of $c=0.001$ eV nm and $d=0.5$ eV nm^2 for an electron with an effective mass $m=0.05$ m_e and frequency $\omega=10\times10^{14}$ Hz in the InGaAs semiconductor.

ground state wave function $\psi_{0l}(r)$ of the CpH potential for the cases $l=0$ and $l=1$, respectively, using the values of potential parameter $c=0.001$ eV nm for an electron with effective mass $m=0.05m_e$ and frequency $\omega\le10\times10^{14}$ Hz. Further, we plot the ground state wave function $\psi_{0l}(r)$ of the CMpH potential for the cases $l=0$ and $l=1$, respectively, using the values of potential parameters $c=0.001$ eV nm and $d=0.5$ eV fm^2 for an electron with effective mass $m=0.05m_e$ and frequency $\omega=10\times10^{14}$ Hz. It is found that the wave functions for the CpH and CMpH are slightly different in shape as the choice of the parameter $d\to0$. In Figs. 4 and 5, we show the variation of electron energy as a function of parameter c in the interval $0.06\le c=0.10$ eV nm taking the parameter value $d=0.01$ eV nm^2, $\omega=8\times10^{14}$ Hz and $m=0.05m_e$ for the cases $l=0$ and $l=1$, respectively. It is seen in

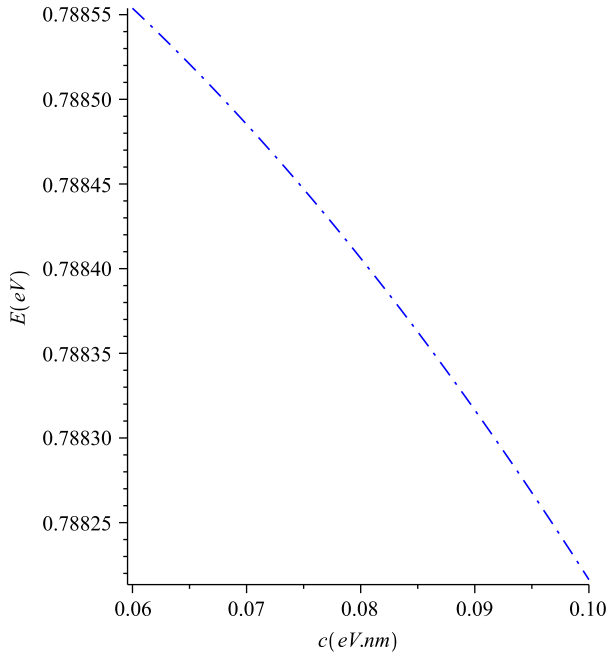

Fig. 4. Ground state energy of electron versus c, for the case $l=0$, $\omega=8\times10^{14}$ Hz and $d=0.01$ eV nm^2.

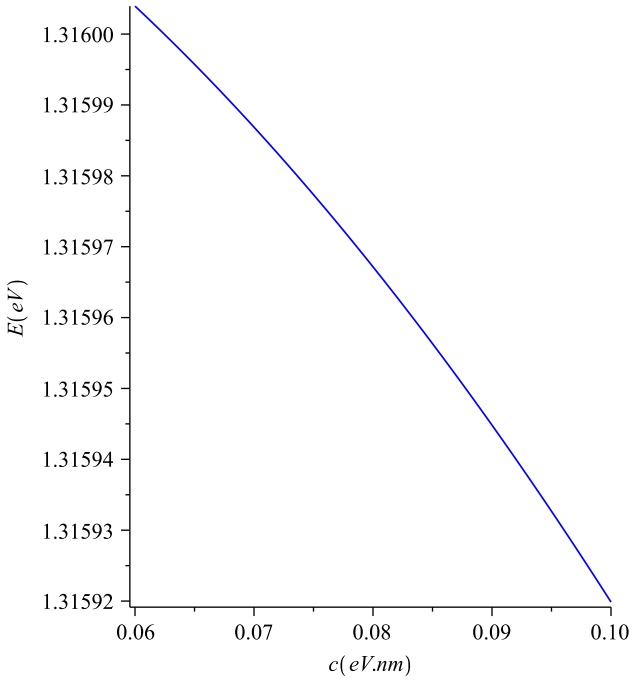

Fig. 5. Ground state energy of electron versus c, for the case $l=1$, $\omega=8\times10^{14}$ Hz and $d=0.01$ eV nm^2.

Figs. 4 and 5 that when the parameter c increases, the electron energy decreasing in the InGaAs semiconductor. In Fig. 6, we plot the first excited state electron energy,

$$E_{1l}=(4+l')\frac{\hbar\omega}{2}-\frac{2(2+l')^2}{(1+l')^2(3+l')^2}\frac{mc^2}{\hbar^2}$$

$$\times\left[1+\sqrt{1+\frac{(1+l')(3+l')}{(2+l')^2}\left[\frac{\hbar^3\omega}{2mc^2}(1+l')-1\right]}\right]^2. \quad (41)$$

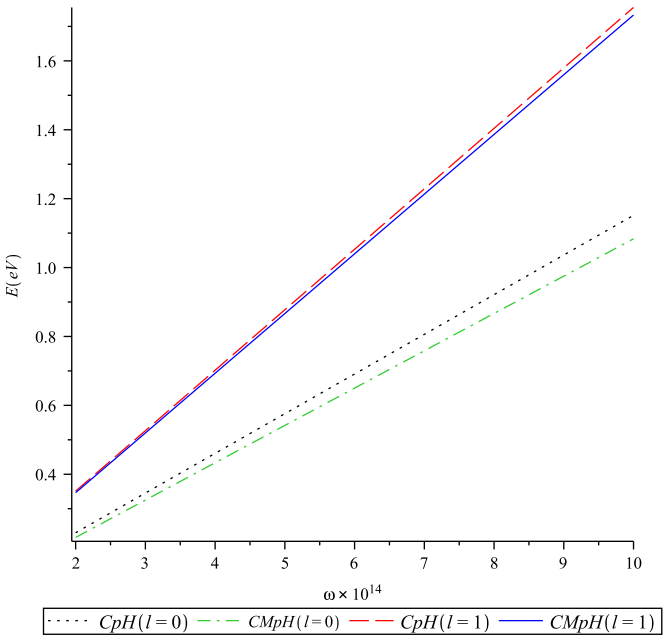

Fig. 6. Comparison of the first excited state electron energy in InGaAs semiconductor versus ω in the field of the CpH and CMpH potentials with $c=0.001$ eV nm and $d=0.5$ eV nm^2 for the cases $l=0$ and $l=1$, respectively.

Table 1
Lowest ($n=0$) energy spectra (for $\hbar=m=1$).

Parameter values					$-E_{0l}$		
a	b	c	d[a]	l	*Numerical*	*Present*	SSUSYQM [25]
$\frac{1}{32}$	1	4	0	0	7.618	7.625	7.625
$\frac{1}{32}$	1	8	0	1	7.368	7.375	7.375
$\frac{1}{32}$	1	12	0	2	7.120	7.125	7.125

[a] We have taken $d=0$ for the sake of comparison.

versus ω in the interval $2\times10^{14}\le\omega\le10\times10^{14}$ Hz taking the value of $c=0.001$ eV fm and $d=0$ eV fm^2 for the cases $l=0$ and $l=1$, respectively. Also we take the value of the parameter $d=0.5$ eV fm^2. We remark that the strongly attractive singular part $-d/r^2$ resembles the centrifugal barrier term $l(l+1)/r^2$ in the Schrödinger equation. This attractive term $-d/r^2$ together with the harmonic oscillator part ar^2 constitute the so-called PHO when $b=0$ ($\beta=0$) in Eq. (14b) leading to $c=0$ in Eq. (14c).

In Table 1, we calculate the lowest ($n=0$) energy states ($l=0,1,2$) from Eq. (36) and from the numerical solution of the RSE (7) taking the parameter values given by Ref. [25] utilized the supersymmetry quantum mechanics (SUSYQM). It is clear that the calculated energy states in the present work are in good agreement with the results obtained numerically via SUSYQM [25]. The accuracy of our numerical results is 0.0070–0.0095%.

4. Conclusion

In this work, we explored the analytical exact solution for the energy eigenvalues and their associated wave functions of a particle in the field of Cornell-modified plus harmonic confining potential. We have used the analytical exact iteration method (AEIM) which required making a trial ansatz for the wave function. The general equation for the energy eigenvalues is given by Eq. (36) with some restrictions on the potential parameters.

If one takes $b=0$ providing $c=0$, for which the potential (1) turns to become the PHO potential with energy eigenvalues:

$$E_{nl}=\sqrt{\frac{\hbar^2 a}{2m}}\left(2+2n+\sqrt{(2l+1)^2-\frac{8md}{\hbar^2}}\right) \tag{42}$$

and the corresponding wave functions

$$\psi_{nl}(r)=N_l\prod_{i=1}^{n}\left(r-\alpha_i^{(n)}\right)r^{(-1+l')/2}\exp\left(-\frac{1}{2}\sqrt{\frac{2ma}{\hbar^2}}r^2\right) \tag{43}$$

The present results in Eqs. (42) and (43) coincide with Eqs. (15) and (16) of Ref. [26] obtained by the exact polynomial method, Eqs. (72) and (78) of Ref. [27] obtained by the Nikiforov–Uvarov method and Eqs. (30) and (31) of Ref. [28] obtained by the wave function ansatz method after setting $D_0/r_0^2=a$, $D_0 r_0^2=-d_0$ and $2D_0=0$. The model solved in the present work can be used in modeling the perturbed field of PHO or electron confined in spherical quantum dots [11,13,14]. Also, our solution to this confining potential is being considered important in many different fields of physics, such as atomic and molecular physics [29,30], particle physics [31,32], plasma physics and solid-state physics [33–37].

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