Relativistic symmetries in the Rosen–Morse potential and tensor interaction using the Nikiforov–Uvarov method*

Sameer M Ikhdair[a)b)] and Majid Hamzavi[c)†]

[a)] Department of Electrical and Electronic Engineering, Near East University, 922022 Nicosia, Northern Cyprus, Mersin 10, Turkey
[b)] Department of Physics, Faculty of Science, An-Najah National University, Nablus, West Bank, Palestine
[c)] Department of Science and Engineering, Abhar Branch, Islamic Azad University, Abhar, Iran

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Approximate analytical bound-state solutions of the Dirac particle in the fields of attractive and repulsive Rosen–Morse (RM) potentials including the Coulomb-like tensor (CLT) potential are obtained for arbitrary spin–orbit quantum number κ. The Pekeris approximation is used to deal with the spin–orbit coupling terms $\kappa(\kappa\pm1)r^{-2}$. In the presence of exact spin and pseudospin (p-spin) symmetries, the energy eigenvalues and the corresponding normalized two-component wave functions are found by using the parametric generalization of the Nikiforov–Uvarov (NU) method. The numerical results show that the CLT interaction removes degeneracies between the spin and p-spin state doublets.

Keywords: Dirac equation, RM potential, CLT potential, spin and p-spin symmetries, NU method, approximation schemes

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1. Introduction

In the framework of the Dirac equation, the spin symmetry occurs when the difference between the scalar $S(r)$ and vector $V(r)$ potentials is a constant, i.e., $\Delta(r)=C_{\rm s}$ and the p-spin symmetry occurs when the sum of the scalar and vector potentials is a constant, i.e., $\Sigma(r)=C_{\rm ps}$.[1–3] The spin symmetry is relevant to mesons.[4] The p-spin symmetry concept has been applied to many systems in nuclear physics and relevant areas[2–7] and is used to explain the features of deformed nuclei,[8] the super-deformation[9] and to establish an effective nuclear shell-model scheme.[5,6,10] The p-spin symmetry introduced in nuclear theory refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n,l,j=l+1/2)$ and $(n-1, l+2, j=l+3/2)$, where n, l, and j are the single-nucleon radial, orbital, and total angular momentum quantum numbers for a single particle, respectively.[5,6] The total angular momentum is given as $j=\tilde{l}+\tilde{s}$, where $\tilde{l}=l+1$ is the p-angular momentum and $\tilde{s}=1/2$ is the p-spin angular momentum. In real nuclei, the p-spin symmetry is only an approximation, and the extent of approximation depends on the p-centrifugal potential and p-spin orbital potential.[11] Alhaidari *et al.*[12] investigated in detail the physical interpretation on the three-dimensional Dirac equation in the context of spin symmetry limitation $\Delta(r)=0$ and p-spin symmetry limitation $\Sigma(r)=0$.

The exact solutions of the Dirac equation for the exponential-type potentials are possible only for the s-wave state ($l=0$ case). However, for l-states an approximation scheme needs to be used to deal with the centrifugal and p-centrifugal terms. Many authors have used different methods to study the partially exactly solvable and exactly solvable Schrödinger, Klein–Gordon (KG), Dirac, and semi-relativistic Salpeter equations in one-, three-, and arbitrary dimensions for different potentials.[13–25] In the context of a spatially-dependent mass, we have also used a proposed approximation scheme[26,27] for the centrifugal term to find a quasi-exact analytic bound-state solution of the radial KG equation with spatially-dependent effective mass for scalar and vector Hulthén potentials in any arbitrary dimension and orbital angular momentum quantum number l within the framework of the Nikiforov–Uvarov (NU) method.[26–29] For more information, one can see Refs. [30]–[54].

Another physical potential is the Rosen–Morse (RM) potential[55] expressed in the form

$$V(r)=-V_1\,\mathrm{sech}^2\,\alpha r+V_2\tanh\alpha r, \tag{1}$$

where V_1 and V_2 denote the depths of the potential and α is the range of the potential. The RM potential is useful for describing the interatomic interaction of the linear molecules and is helpful for discussing polyatomic vibration energies such as the vibration states of the NH_3 molecule.[55] It is shown that the RM potential and its PT-symmetric version are the special cases of the five-parameter exponential-type potential model.[56,57] The exact energy spectrum of the trigonometric RM potential has been investigated by using supersymmetric (SUSY) and improved quantization rule methods.[58,59]

Lisboa *et al.*[60] have studied a generalized relativistic harmonic oscillator for spin-1/2 fermions by solving the Dirac equation with quadratic vector and scalar potentials including

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†Corresponding author. E-mail: majid.hamzavi@gmail.com

a linear tensor potential with spin and p-spin symmetry. Further, Akçay[61] has shown that the Dirac equation for scalar and vector quadratic potentials and the Coulomb-like tensor potential with spin and p-spin symmetry can be solved exactly. In these investigations, it has been found that the tensor interaction removes the degeneracy between two states in the p-spin doublets. The tensor coupling under the spin and p-spin symmetry has also been studied in Refs. [62] and [63]. Further, the nuclear properties have been studied by using tensor couplings.[64,65] Very recently, various types of potentials like Hulthén[66] and Woods–Saxon[67] including the Coulomb-like potential have been studied with the conditions of spin and p-spin symmetry. The p-spin symmetric solution of the Dirac–Morse problem with the Coulomb-like tensor potential for any spin–orbit quantum number κ has been studied.[68]

In the present work, our aim is to present the analytical solutions of the Dirac–RM problem with the Coulomb-like tensor potential for arbitrary spin–orbit quantum number κ that is not large and vibrations of the small amplitude about the minimum. This approximation has been introduced by Lu.[69] and used in Ref. [70] to deal with the centrifugal term near the minimum point $r = r_e$.

The rest of the present paper is structured as follows. In Section 2, we outline the NU method. In Section 3, we obtain the analytical spin and p-spin symmetric bound state solutions of the (3+1)-dimensional Dirac equation for the RM potential model including the Coulomb-like tensor (CLT) potential by means of the NU method. Finally, the relevant conclusion is given in Section 4.

2. Parametric generalization of the NU method

The NU method is used to solve second order differential equations with an appropriate coordinate transformation $s = s(r)$[28]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, mostly second degree ones, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without needing to check the validity of solution, we present a shortcut for the method. So, at first we write the general form of the Schrödinger-like equation (2) in a more general form applicable to any potential as follows:[71]

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(c_3 - c_4 s)}\right)\psi_n'(s) + \left(\frac{-As^2 + Bs - C}{s^2(c_3 - c_4 s)^2}\right)\psi_n(s) = 0, \tag{3}$$

satisfying the wave functions

$$\psi_n(s) = \phi(s) y_n(s). \tag{4}$$

Comparing Eq. (3) with its counterpart Eq. (2), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(c_3 - c_4 s),$$

$$\tilde{\sigma}(s) = -As^2 + Bs - C, \tag{5}$$

where c_i $(i = 1,2,3,4)$, A, B, and C are constant coefficients to be found for the potential model. Following the NU method,[28] we obtain the following,[71]

(i) the relevant constants

$$\begin{aligned}
&c_5 = \frac{1}{2}(c_3 - c_1), && c_6 = \frac{1}{2}(c_2 - 2c_4),\\
&c_7 = c_6^2 + A, && c_8 = 2c_5c_6 - B,\\
&c_9 = c_5^2 + C, && c_{10} = c_4(c_3c_8 + c_4c_9) + c_3^2c_7,\\
&c_{11} = \frac{2}{c_3}\sqrt{c_9},\ c_3 \neq 0, && c_{12} = \frac{2}{c_3c_4}\sqrt{c_{10}},\ c_4 \neq 0,\\
&c_{13} = \frac{1}{c_3}(c_5 + \sqrt{c_9}), && c_{14} = \frac{1}{c_3c_4}(\sqrt{c_{10}} - c_4c_5 - c_3c_6),\\
&c_{15} = \frac{2}{c_3}\sqrt{c_{10}}, && c_{16} = \frac{1}{c_3}(\sqrt{c_{10}} - c_4c_5 - c_3c_6);
\end{aligned} \tag{6}$$

(ii) the essential polynomial functions:

$$\pi(s) = c_5 + \sqrt{c_9} - \frac{1}{c_3}(c_4\sqrt{c_9} + \sqrt{c_{10}} - c_3c_6)s, \tag{7}$$

$$k = -\frac{1}{c_3^2}(c_3c_8 + 2c_4c_9 + 2\sqrt{c_9c_{10}}), \tag{8}$$

$$\tau(s) = c_3 + 2\sqrt{c_9} - \frac{2}{c_3}(c_3c_4 + c_4\sqrt{c_9} + \sqrt{c_{10}})s, \tag{9}$$

$$\tau'(s) = -\frac{2}{c_3}(c_3c_4 + c_4\sqrt{c_9} + \sqrt{c_{10}}) < 0; \tag{10}$$

(iii) the energy equation:

$$\begin{aligned}
c_2n \ - \ &(2n+1)c_6 + \frac{1}{c_3}(2n+1)(\sqrt{c_{10}} + c_4\sqrt{c_9})\\
&+ \ n(n-1)c_41 + \frac{1}{c_3^2}(c_3c_8 + 2c_4c_9 + 2\sqrt{c_9c_{10}}) = 0;
\end{aligned} \tag{11}$$

(iv) the wave functions

$$\rho(s) = s^{c_{11}}(c_3 - c_4s)^{c_{12}}, \tag{12}$$

$$\varphi(s) = s^{c_{13}}(c_3 - c_4s)^{c_{14}},\ c_{13} > 0,\ c_{14} > 0, \tag{13}$$

$$\begin{aligned}
y_n(s) &= P_n^{(c_{11},c_{12})}(c_3 - 2c_4s),\ c_{11} > -1,\ c_{12} > -1,\\
&s \in [(c_3 - 1)/2c_4, (c_3 + 1)/2c_4],
\end{aligned} \tag{14}$$

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{13}}(c_3 - c_4s)^{c_{14}}P_n^{(c_{11},c_{12})}(c_3 - 2c_4s), \tag{15}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1,1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1-2s) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; s), \tag{16}$$

and $N_{n\kappa}$ is a normalization constant. When $c_4 = 0$, the Jacobi polynomial turns into the generalized Laguerre polynomial and the constants relevant to this polynomial change are

$$\begin{aligned}
&\lim_{c_4\to 0} P_n^{(c_{11},c_{12})}(c_3 - 2c_4s) = L_n^{c_{11}}(c_{15}s),\\
&\lim_{c_4\to 0}(c_3 - c_4s)^{c_{14}} = \exp(-c_{16}s),\\
&\psi_{n\kappa}(s) = N_{n\kappa}\exp(-c_{16}s)L_n^{c_{11}}(c_{15}s),
\end{aligned} \tag{17}$$

where $L_n^{c_{11}}(c_{15}s)$ are the generalized Laguerre polynomials and $N_{n\kappa}$ is a normalization constant.

3. Dirac bound states of the RM potential and CLT potential

The Dirac equation for a particle of mass m moving in the field of attractive radial scalar $S(r)$, repulsive vector $V(r)$, and tensor $U(r)$ potentials (in the relativistic units $\hbar = c = 1$) takes the form[35]

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(m+S(r))+V(r)-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}), \tag{18}$$

with E being the relativistic energy of the system, $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$ is the three-dimensional (3D) momentum operator and $\boldsymbol{\alpha}$ and β represent the 4×4 usual Dirac matrices which are expressed in terms of the three 2×2 Pauli matrices and the 2×2 unit matrix, respectively. For spherical nuclei, the Dirac Hamiltonian commutes with the total angular momentum operator $\boldsymbol{J} = \boldsymbol{L}+\boldsymbol{S}$ and the spin–orbit coupling operator $\boldsymbol{K} = -\beta(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ where $\boldsymbol{L}$ and $\boldsymbol{S}$ are the orbital momentum and the spin momentum, respectively. The eigenvalues of the spin–orbit coupling operator are $\kappa = l > 0$ and $\kappa = -(l+1) < 0$ for unaligned spin $(j = l-1/2)$ and aligned $(j = l+1/2)$, respectively. Thus, the Dirac wave function takes the form

$$\psi_{n\kappa}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)\mathrm{Y}^{l}_{jm}(\theta,\varphi) \\ \mathrm{i}G_{n\kappa}(r)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi)\end{pmatrix}, \tag{19}$$

where $\mathrm{Y}^{l}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi)$ are the spin and p-spin spherical harmonics, respectively. $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the upper- and lower-spinor radial functions, respectively. Inserting Eq. (19) into Eq. (18) and using the relationship [72]

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \tag{20a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \boldsymbol{\sigma}\cdot\hat{r}\Big(\hat{r}\cdot\boldsymbol{p}+\mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\Big), \tag{20b}$$

and properties

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi) = (\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi), \tag{21a}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{l}_{jm}(\theta,\varphi) = -(\kappa+1)\mathrm{Y}^{l}_{jm}(\theta,\varphi), \tag{21b}$$

$$\Big(\frac{\boldsymbol{\sigma}\cdot r}{r}\Big)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi) = -\mathrm{Y}^{l}_{jm}(\theta,\varphi), \tag{21c}$$

$$\Big(\frac{\boldsymbol{\sigma}\cdot r}{r}\Big)\mathrm{Y}^{l}_{jm}(\theta,\varphi) = -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi), \tag{21d}$$

we obtain the following two coupled differential equations satisfying the upper and lower radial functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\Big(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\Big)F_{n\kappa}(r) = (m+E-\Delta(r))G_{n\kappa}(r), \tag{22a}$$

$$\Big(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\Big)G_{n\kappa}(r) = (m-E+\Sigma(r))F_{n\kappa}(r), \tag{22b}$$

where the sum and difference potentials are defined as

$$\Sigma(r) = V(r)+S(r) \quad \text{and} \quad \Delta(r) = V(r)-S(r), \tag{23}$$

respectively. Combining Eqs. (22a) and (22b), we obtain the second-order differential equations satisfying the radial functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$, respectively[73,74]

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\Big(\frac{2\kappa}{r}-U(r)-\frac{\mathrm{d}}{\mathrm{d}r}\Big)U(r)-[m+E_{n\kappa}-\Delta(r)][m-E_{n\kappa}+\Sigma(r)] + \frac{1}{m+E_{n\kappa}-\Delta(r)}\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\Big(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\Big)\Big\}F_{n\kappa}(r) = 0, \tag{24}$$

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\Big(\frac{2\kappa}{r}-U(r)+\frac{\mathrm{d}}{\mathrm{d}r}\Big)U(r)-[(m+E_{n\kappa}-\Delta(r))(m-E_{n\kappa}+\Sigma(r))] - \frac{1}{m-E_{n\kappa}+\Sigma(r)}\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\Big(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\Big)\Big\}G_{n\kappa}(r) = 0, \tag{25}$$

where $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$. These radial wave equations are required to satisfy the necessary boundary conditions in the interval $r\in(0,\infty)$, i.e., $F_{n\kappa}(r=0) = G_{n\kappa}(0) = 0$ and $F_{n\kappa}(r\to\infty)\cong0$, $G_{n\kappa}(r\to\infty)\cong0$.

3.1. Spin symmetric limit

Under the exact spin symmetric condition, we take the sum potential, $\Sigma(r)$, as the RM potential model, the difference potential, $\Delta(r)$, as a constant, and the tensor potential, $U(r)$, as CLT potential. Then we have the following forms

$$\Sigma(r) = -4V_1\frac{\mathrm{e}^{-2\alpha r}}{(1+\mathrm{e}^{-2\alpha r})^2}+V_2\frac{(1-\mathrm{e}^{-2\alpha r})}{(1+\mathrm{e}^{-2\alpha r})}, \quad \Delta(r) = C_{\mathrm{s}},\ U(r) = -\frac{H}{r}, \tag{26}$$

where C_{s} and $H = Ze^2/(4\pi\varepsilon_0)$ are two constants.

To find an approximate solution for the radial Dirac equation with the RM potential, we need to use the Pekeris approximation[69,75] for the spin–orbit centrifugal term near the minimum point $r = r_{\mathrm{e}}$ as

$$\frac{1}{r^2}\approx\frac{1}{r_{\mathrm{e}}^2}\Big[D_0+D_1\frac{-\mathrm{e}^{-2\alpha r}}{1+\mathrm{e}^{-2\alpha r}}+D_2\Big(\frac{-\mathrm{e}^{-2\alpha r}}{1+\mathrm{e}^{-2\alpha r}}\Big)^2\Big], \tag{27a}$$

where

$$D_0 = 1-\Big(\frac{1+\mathrm{e}^{-2\alpha r_{\mathrm{e}}}}{2\alpha r_{\mathrm{e}}}\Big)\Big(\frac{8\alpha r_{\mathrm{e}}}{1+\mathrm{e}^{-2\alpha r_{\mathrm{e}}}}-(3+2\alpha r_{\mathrm{e}})\Big),$$

$$D_1 = -2(\mathrm{e}^{2\alpha r_{\mathrm{e}}}+1)\Big[3\Big(\frac{1+\mathrm{e}^{-2\alpha r_{\mathrm{e}}}}{2\alpha r_{\mathrm{e}}}\Big)-(3+2\alpha r_{\mathrm{e}})\Big(\frac{1+\mathrm{e}^{-2\alpha r_{\mathrm{e}}}}{2\alpha r_{\mathrm{e}}}\Big)\Big],$$

$$D_2 = (\mathrm{e}^{2\alpha r_{\mathrm{e}}}+1)^2\Big(\frac{1+\mathrm{e}^{-2\alpha r_{\mathrm{e}}}}{2\alpha r_{\mathrm{e}}}\Big)^2$$

$$\times\left(3+2\alpha r_{\rm e}-\frac{4\alpha r_{\rm e}}{1+{\rm e}^{-2\alpha r_{\rm e}}}\right), \tag{27b}$$

and higher order terms are neglected. Inserting Eqs. (26) and (27) and introducing a new parameter change $z(r)=-{\rm e}^{-2\alpha r}$, we are allowed to decompose the spin-symmetric Dirac equation (24) into the Schrödinger-like equation in the spherical coordinates for the upper-spinor component $F_{n\kappa}(r)$ as follows:

$$\left[\frac{{\rm d}^2}{{\rm d}z^2}+\frac{(1-z)}{z(1-z)}\frac{\rm d}{{\rm d}z}+\frac{(-\beta_1 z^2+\beta_2 z-\varepsilon_{n\kappa}^2)}{z^2(1-z)^2}\right]F_{n\kappa}(z)=0,$$

$$F_{n\kappa}(0)=F_{n\kappa}(1)=0, \tag{28a}$$

$$\varepsilon_{n\kappa}=\frac{1}{2\alpha}\sqrt{\frac{\omega}{r_{\rm e}^2}D_0-\tilde{E}_{n\kappa}^2+\tilde{V}_2}>0, \tag{28b}$$

$$\beta_1=\frac{1}{4\alpha^2}\left[\frac{\omega}{r_{\rm e}^2}(D_0+D_1+D_2)-\tilde{E}_{n\kappa}^2-\tilde{V}_2\right], \tag{28c}$$

$$\beta_2=\frac{1}{4\alpha^2}\left[\frac{\omega}{r_{\rm e}^2}(2D_0+D_1)-2\tilde{E}_{n\kappa}^2-4\tilde{V}_1\right], \tag{28d}$$

where $\tilde{V}_1=V_1(m+E_{n\kappa}-C_{\rm s})$, $\tilde{V}_2=V_2(m+E_{n\kappa}-C_{\rm s})$, $\tilde{E}_{n\kappa}^2=(E_{n\kappa}+m-C_{\rm s})(E_{n\kappa}-m)$, and $\omega=(\kappa+H)(\kappa+H+1)$. Further, the explicit forms of the constant D_i $(i=1,2,3)$ are defined in Refs. [69] and [70] and are expressed in terms of the potential parameters. In order to solve Eq. (28a) by the NU method, we should compare it with Eq. (2) to obtain the following particular values for the parameters:

$$\tilde{\tau}(z)=1-z,\ \sigma(z)=z(1-z),$$
$$\tilde{\sigma}(z)=-\beta_1 z^2+\beta_2 z-\varepsilon_{n\kappa}^2. \tag{29}$$

Comparing Eq. (29) with Eq. (5), we can easily obtain the coefficients c_i $(i=1,2,3,4)$ and the analytical expressions A, B, and C. However, the values of the coefficients c_i $(i=5,6,\ldots,16)$ are found from relationship (6). Therefore, the specific values of the coefficients c_i $(i=1,2,\ldots,16)$ together with A, B, and C are

$$c_1=c_2=c_3=c_4=1,\quad c_5=0,\quad c_6=-\frac{1}{2},$$
$$c_7=\frac{1}{4}+\beta_1,\quad c_8=-\beta_2,\quad c_9=\varepsilon_{n\kappa}^2,\quad c_{10}=\left(\delta+\frac{1}{2}\right)^2,$$
$$c_{11}=2\varepsilon_{n\kappa},\quad c_{12}=c_{15}=2\delta+1,\quad c_{13}=\varepsilon_{n\kappa},$$
$$c_{14}=c_{16}=\delta+1,\quad A=\beta_1,\quad B=\beta_2,\quad C=\varepsilon_{n\kappa}^2,$$
$$\delta=\frac{1}{2}\left(-1+\sqrt{1+\frac{1}{\alpha^2}\left(\frac{\omega D_2}{r_{\rm e}^2}+4\tilde{V}_1\right)}\right). \tag{30}$$

From Eqs. (7)–(9) together with the coefficients given in Eq. (30), we can calculate the essential parameters $\pi(z)$, k, and $\tau(z)$ as

$$\pi(z)=\varepsilon_{n\kappa}-(1+\varepsilon_{n\kappa}+\delta)z, \tag{31}$$
$$k=\beta_2-[2\varepsilon_{n\kappa}^2+(2\delta+1)\varepsilon_{n\kappa}], \tag{32}$$

and

$$\tau(z)=1+2\varepsilon_{n\kappa}-(3+2\varepsilon_{n\kappa}+2\delta)z,$$
$$\tau'(z)=-(3+2\varepsilon_{n\kappa}+2\delta)<0, \tag{33}$$

with prime denoting the derivative with respect to z. Equation (11) gives the energy equation for the RM potential including the CLT potential in the Dirac theory as

$$(m+E_{n\kappa}-C_{\rm s})(m-E_{n\kappa}+V_2)$$
$$=-\frac{\omega D_0}{r_{\rm e}^2}+\alpha^2\left[\frac{-2V_1(m+E_{n\kappa}-C_{\rm s})+\omega(D_1+D_2)/r_{\rm e}^2}{4\alpha^2(n+\delta+1)}\right.$$
$$\left.-(n+\delta+1)\right]^2. \tag{34}$$

Further, for the exact spin symmetric case, $S(\boldsymbol{r})=V(\boldsymbol{r})$ or $C_{\rm s}=0$, we obtain

$$(m+E_{n\kappa})(m-E_{n\kappa}+V_2)$$
$$=-\frac{\omega D_0}{r_{\rm e}^2}+\alpha^2\left[\frac{-2V_1(m+E_{n\kappa})+\omega(D_1+D_2)/r_{\rm e}^2}{4\alpha^2(n+\tilde{\delta}+1)}\right.$$
$$\left.-(n+\tilde{\delta}+1)\right]^2,$$

with

$$\tilde{\delta}=\delta(C_{\rm s}\to 0)$$
$$=\frac{1}{2}\left(-1+\sqrt{1+\frac{1}{\alpha^2}\left(\frac{\omega D_2}{r_{\rm e}^2}+4V_1(m+E_{n\kappa})\right)}\right).$$

We now find the corresponding wave functions for this potential model. Referring to Eqs. (12), (13), and (30), we find the functions

$$\rho(z)=z^{2\varepsilon_{n\kappa}}(1-z)^{2\delta+1}, \tag{35}$$
$$\varphi(z)=z^{\varepsilon_{n\kappa}}(1-z)^{\delta+1},\ \varepsilon_{n\kappa}>0,\ \delta>0. \tag{36}$$

Hence, equation (14) gives

$$y_n(z)=A_n z^{-2\varepsilon_{n\kappa}}(1-z)^{-(2\delta+1)}\frac{{\rm d}^n}{{\rm d}z^n}[z^{n+2\varepsilon_{n\kappa}}(1-z)^{n+2\delta+1}]$$
$$={\rm P}_n^{(2\varepsilon_{n\kappa},2\delta+1)}(1-2z),$$
$$2\varepsilon_{n\kappa}>-1,\ 2\delta+1>-1,\ z\in[0,1], \tag{37}$$

where ${\rm P}_n^{(\mu,\nu)}(x)$ is the Jacobi polynomials with $\mu>-1$, $\nu>-1$, and $x\in[-1,+1]$. By using $F_{n\kappa}(z)=\varphi(z)y_n(z)$, we obtain the radial upper-spinor wave functions from Eq. (15) as

$$F_{n\kappa}(r)=N_{n\kappa}(-{\rm e}^{-2\alpha r})^{\varepsilon_{n\kappa}}(1+{\rm e}^{-2\alpha r})^{\delta+1}{\rm P}_n^{(2\varepsilon_{n\kappa},2\delta+1)}(1+2{\rm e}^{-2\alpha r})$$
$$=N_{n\kappa}({\rm e}^{-2\alpha r})^{\varepsilon_{n\kappa}}(1+{\rm e}^{-2\alpha r})^{\delta+1}{}_2{\rm F}_1(-n,n+2(\varepsilon_{n\kappa}+\delta+1);\ 2\varepsilon_{n\kappa}+1;\ -{\rm e}^{-2\alpha r}), \tag{38}$$

where the normalization constant has been calculated in Ref. [70]. The lower component $G_{n\kappa}(r)$ can be obtained as follows:[70]

$$G_{n\kappa}(r)=N_{n\kappa}\frac{(-{\rm e}^{-2\alpha r})^{\varepsilon_{n\kappa}}(1+{\rm e}^{-2\alpha r})^{\delta+1}}{(m+E_{n\kappa}-C_{\rm s})}\left[-2\alpha\varepsilon_{n\kappa}-\frac{2\alpha(\delta+1){\rm e}^{-2\alpha r}}{(1+{\rm e}^{-2\alpha r})}+\frac{\kappa}{r}\right]$$
$$\times{}_2{\rm F}_1(-n,n+2(\varepsilon_{n\kappa}+\delta+1);2\varepsilon_{n\kappa}+1;-{\rm e}^{-2\alpha r})$$

$$+N_{n\kappa}\Big[\frac{2\alpha n[n+2(\varepsilon_{n\kappa}+\delta+1)](-\mathrm{e}^{-2\alpha r})^{\varepsilon_{n\kappa}+1}(1+\mathrm{e}^{-2\alpha r})^{\delta+1}}{(2\varepsilon_{n\kappa}+1)(m+E_{n\kappa}-C_{\rm s})}\Big]$$
$$\times {}_2\mathrm{F}_1(-n+1,n+2(\varepsilon_{n\kappa}+\delta+3/2);2(\varepsilon_{n\kappa}+1);-\mathrm{e}^{-2\alpha r}), \quad (39)$$

where $E_{n\kappa}\neq -m$ for exact spin symmetry. Hence, the spin symmetric solution has only a positive energy spectrum. Here, it should be noted that the hypergeometric series ${}_2\mathrm{F}_1(-n,n+2(\varepsilon_{n\kappa}+\delta+1);2\varepsilon_{n\kappa}+1;-\mathrm{e}^{-2\alpha r})$ terminates for the $n=0$ state and thus does not diverge for all values of real parameters δ and $\varepsilon_{n\kappa}$.

3.2. The P-spin symmetric limit

Inserting $\Delta(r)$ as the RM potential, $\Sigma(r)$ as a constant, and $U(r)$ as a Coulomb-like tensor potential, i.e.,

$$\Delta(r) = -4V_1\frac{\mathrm{e}^{-2\alpha r}}{(1+\mathrm{e}^{-2\alpha r})^2}+V_2\frac{(1-\mathrm{e}^{-2\alpha r})}{(1+\mathrm{e}^{-2\alpha r})},$$
$$\Sigma(r) = C_{\rm ps},\ U(r)=-\frac{H}{r}, \quad (40)$$

into Eq. (25), we obtain the following Schrödinger-like equation for the lower-spinor component $G_{n\kappa}(r)$:

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(1-z)}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{(-\bar\beta_1 z^2+\bar\beta_2 z-\bar\varepsilon_{n\kappa}^2)}{z^2(1-z)^2}\Big]G_{n\kappa}(z)=0, \quad (41a)$$

$$\bar\varepsilon_{n\kappa}=\frac{1}{2\alpha}\sqrt{\frac{\bar\omega}{r_{\rm e}^2}D_0-\bar E_{n\kappa}^2+\bar V_2}>0, \quad (41b)$$

$$\bar\beta_1=\frac{1}{4\alpha^2}\Big[\frac{\bar\omega}{r_{\rm e}^2}(D_0+D_1+D_2)-\bar E_{n\kappa}^2-\bar V_2\Big], \quad (41c)$$

$$\bar\beta_2=\frac{1}{4\alpha^2}\Big[\frac{\bar\omega}{r_{\rm e}^2}(2D_0+D_1)-2\bar E_{n\kappa}^2-4\bar V_1\Big], \quad (41d)$$

where $\bar V_1=V_1(E_{n\kappa}-m-C_{\rm ps})$, $\bar V_2=V_2(E_{n\kappa}-m-C_{\rm ps})$, $\bar E_{n\kappa}^2=(E_{n\kappa}-m-C_{\rm ps})(E_{n\kappa}+m)$, and $\bar\omega=(\kappa+H)(\kappa+H-1)$.

To avoid repetition in the solution of Eq. (41a), a first inspection for the relationship between the present set of parameters ($\bar\varepsilon_{n\kappa}$, $\bar\beta_1$, $\bar\beta_2$) and the previous set ($\varepsilon_{n\kappa}$, β_1, β_2) indicates that the negative energy solution for p-spin symmetry, where $S(r)=-V(r)$, can be obtained directly from the above positive energy solution for spin symmetry by using parameter mapping as follows:[70]

$$F_{n\kappa}(r)\to G_{n\kappa}(r);\ V(r)\to -V(r)(V_1\to -V_1, V_2\to -V_2);$$
$$E_{n\kappa}\to -E_{n\kappa};\ C_{\rm s}\to -C_{\rm ps}. \quad (42)$$

Following the previous results with the above transformations, we finally arrive at the energy equation

$$(m-E_{n\kappa}+C_{\rm ps})(m+E_{n\kappa}-V_2)$$
$$= -\frac{\bar\omega D_0}{r_{\rm e}^2}+\alpha^2\Big[\frac{2V_1(m-E_{n\kappa}+C_{\rm ps})+\bar\omega(D_1+D_2)/r_{\rm e}^2}{4\alpha^2(n+\eta+1)}$$
$$-(n+\eta+1)\Big]^2, \quad (43)$$

with

$$\eta=\frac{1}{2}\Big(-1+\sqrt{1+\frac{1}{\alpha^2}\Big(\frac{\bar\omega D_2}{r_{\rm e}^2}+4\bar V_1\Big)}\Big). \quad (44)$$

By using $G_{n\kappa}(r)=\varphi(z)y_n(z)$, we obtain the radial lower-spinor wave functions as

$$G_{n\kappa}(r) = \tilde N_{n\kappa}(-\mathrm{e}^{-2\alpha r})^{\bar\varepsilon_{n\kappa}}(1+\mathrm{e}^{-2\alpha r})^{\eta+1}$$
$$\times \mathrm{P}_n^{(2\bar\varepsilon_{n\kappa},2\eta+1)}(1+2\mathrm{e}^{-2\alpha r}), \quad (45)$$

where $G_{n\kappa}(r)$ satisfies the restriction condition for the bound states, i.e., $\eta>0$ and $\bar\varepsilon_{n\kappa}>0$. The normalization constants $\tilde N_{n\kappa}$ are calculated in Ref. [70].

To check our analytical expressions, we calculate the energy levels for the p-spin and spin cases in Tables 1 and 2, respectively using the following set of parameter values:

$m=1.0\ \mathrm{fm}^{-1}$ ($1\ \mathrm{fm}=10^{-15}\ \mathrm{m}$), $V_1=1.0$ fm,
$V_2=-1.0\ \mathrm{fm}^{-1}$, $r_{\rm e}=2.197224577$ fm, $\alpha=0.25\ \mathrm{fm}^{-1}$,
$C_{\rm s}=0\ \mathrm{fm}^{-1}$, and $C_{\rm ps}=-6.0\ \mathrm{fm}^{-1}$.

In Table 1, we observe the degeneracy in the following doublets ($1\mathrm{s}_{1/2}$, $0\mathrm{d}_{3/2}$), ($1\mathrm{p}_{3/2}$, $0\mathrm{f}_{5/2}$), ($1\mathrm{d}_{5/2}$, $0\mathrm{g}_{7/2}$), ($1\mathrm{f}_{7/2}$, $0\mathrm{h}_{9/2}$), and so on. Thus, each pair is considered as a p-spin doublet and has negative energy. In Table 2, we present the energy spectrum for the spin symmetric case. Obviously, the pairs ($n\mathrm{p}_{1/2}$, $n\mathrm{p}_{3/2}$), ($n\mathrm{d}_{3/2}$, $n\mathrm{d}_{5/2}$), ($n\mathrm{f}_{5/2}$, $n\mathrm{f}_{7/2}$), ($n\mathrm{g}_{7/2}$, $n\mathrm{g}_{9/2}$), and so forth are degenerate states. Thus, each pair is considered as a spin doublet and has positive energy. The numerical results in Tables 1 and 2 emphasize that the presence of the CLT interaction removes degeneracies between spin and p-spin state doublets.

Table 1. Energy levels in units of fm^{-1} of the p-spin symmetry RM potential for several values of n and κ with $H=0.5$.

$\tilde l$	$n,\kappa<0$	(l,j)	$E_{n,\kappa<0}$ $H\neq 0$	$E_{n,\kappa<0}$ $H=0$	$n-1,\kappa>0$	$(l+2,j+1)$	$E_{n-1,\kappa>0}$ $H\neq 0$	$E_{n-1,\kappa>0}$ $H=0$
1	1, −1	$1\mathrm{s}_{1/2}$	−1.903134794	−1.738772757	0, 2	$0\mathrm{d}_{3/2}$	−1.538608678	−1.738772757
2	1, −2	$1\mathrm{p}_{3/2}$	−1.538608678	−1.313563183 ,	0, 3	$0\mathrm{f}_{5/2}$	−1.071543282	−1.313563183
3	1, −3	$1\mathrm{d}_{5/2}$	−1.071543282	−0.8177171059	0, 4	$0\mathrm{g}_{7/2}$	−0.5554514514	−0.8177171059
4	1, −4	$1\mathrm{f}_{7/2}$	−0.5554514514	−0.2869876340	0, 5	$0\mathrm{h}_{9/2}$	−0.01385847616	−0.2869876340
1	2, −1	$2\mathrm{s}_{1/2}$	−1.921760586	−1.745387730	1, 2	$1\mathrm{d}_{3/2}$	−1.521903111	−1.745387730
2	2, −2	$2\mathrm{p}_{3/2}$	−1.521903111	−1.273143731	1, 3	$1\mathrm{f}_{5/2}$	−1.01005856	−1.273143731
3	2, −3	$2\mathrm{d}_{5/2}$	−1.010058564	−0.7382341146	1, 4	$1\mathrm{g}_{7/2}$	−0.4607283791	−0.7382341146
4	2, −4	$2\mathrm{f}_{7/2}$	−0.4607283791	−0.1793345097	1, 5	$1\mathrm{h}_{9/2}$	0.1048326489	−0.1793345097

Table 2. Energy levels in units of fm^{-1} of the spin symmetry RM potential for several values of n and κ with $H = 0.5$.

l	$n, \kappa < 0$	$(l, j = l + 1/2)$	$E_{n,\kappa<0}$ $H \neq 0$	$E_{n,\kappa<0}$ $H = 0$	$n, \kappa > 0$	$(l, j = l - 1/2)$	$E_{n,\kappa>0}$ $H \neq 0$	$E_{n,\kappa>0}$ $H = 0$
1	0, −2	$0p_{3/2}$	0.1483955852	0.3935828782	0, 1	$0p_{1/2}$	0.6582373104	0.3935828782
2	0, −3	$0d_{5/2}$	0.6582373104	0.9333946490	0, 2	$0d_{3/2}$	1.214682872	0.9333946490
3	0, −4	$0f_{7/2}$	1.214682872	1.499799551	0, 3	$0f_{5/2}$	1.787442858	1.499799551
4	0, −5	$0g_{9/2}$	1.787442858	2.076830625	0, 4	$0g_{7/2}$	2.367468347	2.076830625
1	1, −2	$1p_{3/2}$	0.2663841239	0.5510806494	1, 1	$1p_{1/2}$	0.8381866907	0.5510806494
2	1, −3	$1d_{5/2}$	0.8381866907	1.127479271	1, 2	$1d_{3/2}$	1.418299684	1.127479271
3	1, −4	$1f_{7/2}$	1.418299684	1.710188317	1, 3	$1f_{5/2}$	2.002843683	1.710188317
4	1, −5	$1g_{9/2}$	2.002843683	2.296064965	1, 4	$1g_{7/2}$	2.589714652	2.296064965

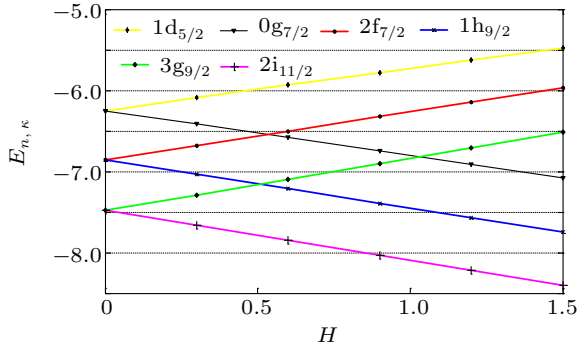

Fig. 1. (color online) The energy levels versus CLT strength H in the p-spin symmetric case using the parameters $m = 1.0$ fm^{-1}, $C_{\mathrm{ps}} = -6.0$ fm^{-1}, $\alpha = 0.25$ fm^{-1}, $r_{\mathrm{e}} = 2.197224577$ fm, $V_1 = 1.0$ fm^{-1}, and $V_2 = -1.0$ fm^{-1}.

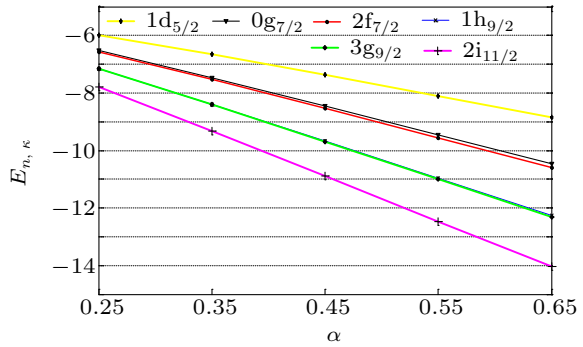

Fig. 2. (color online) Energy levels versus screening parameter α in the p-spin symmetric case using the parameters $m = 1.0$ fm^{-1}, $C_{\mathrm{ps}} = -6.0$ fm^{-1}, $H = 0.5$, $r_{\mathrm{e}} = 2.197224577$ fm, $V_1 = 1.0$ fm^{-1}, and $V_2 = -1.0$ fm^{-1}.

Finally, we plot the relativistic energy eigenvalues of the RM potential and CLT potential with spin and p-spin symmetry limitations in Figs. 1–4. Figures 1 and 2 show the variations of the energy level versus the Coulomb tensor strength H and the screening parameter α respectively in the case of p-spin symmetry considering the following pairs of orbital states $(1d_{5/2}, 0g_{7/2})$, $(2f_{7/2}, 1h_{9/2})$, $(3g_{9/2}, 2i_{11/2})$. From Fig. 1, we observe that in the case of $H = 0$ (no tensor interaction), members of the p-spin doublets have the same energy. However, in the presence of the tensor potential $H \neq 0$, these degeneracies are removed. We can also see in Fig. 1 that p-spin doublet splitting increases with H increasing. The reason for this is that term $2\kappa H$ gives different contributions to each level in the spin doublet because H takes different values for each state in the spin doublet. In Fig. 2, the contribution of the screening parameter α to the p-spin doublet splitting is presented. It can be seen that the magnitude of the energy difference between members of the p-spin doublet decreases as α increases.

Furthermore, in Figs. 3 and 4, we investigate the effects of Coulomb tensor strength H and the screening parameter α on the spin doublet splitting by considering the following orbital pairs: $(1p_{1/2}, 1p_{3/2})$, $(1d_{3/2}, 1d_{5/2})$, and $(1f_{5/2}, 1f_{7/2})$, and one can observe that the results obtained in the spin symmetric limit resemble the ones observed in the p-spin symmetric limit.

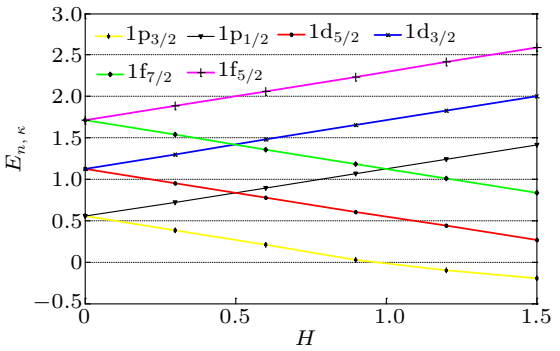

Fig. 3. (color online) Energy levels versus CLT strength H in the spin symmetric case using the parameters $m = 1.0$ fm^{-1}, $C_{\mathrm{s}} = 0$ fm^{-1}, $\alpha = 0.25$ fm^{-1}, $r_{\mathrm{e}} = 2.197224577$ fm, $V_1 = 1.0$ fm^{-1}, and $V_2 = -1.0$ fm^{-1}.

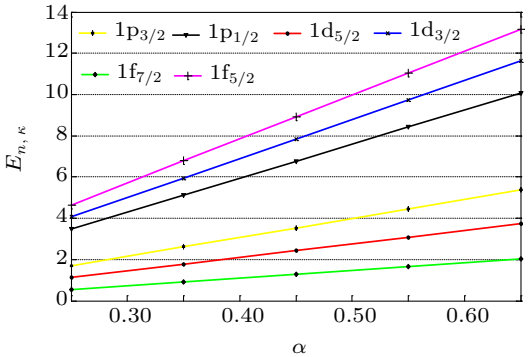

Fig. 4. (color online) Energy levels versus screening parameter α in the spin symmetric case using the parameters $m = 1.0$ fm^{-1}, $C_{\mathrm{s}} = 0$ fm^{-1}, $H = 5.0$, $r_{\mathrm{e}} = 2.197224577$ fm, $V_1 = 1.0$ fm^{-1}, and $V_2 = -1.0$ fm^{-1}.

4. Conclusion

We have obtained analytically the spin and p-spin symmetric energy eigenvalues and the corresponding wave func-

tions of the Dirac–RM problem with CLT potential in the frame of the NU method. For any spin–orbit quantum number κ, we have also found the approximate expressions for the energy eigenvalues and associated wave functions in the closed form. The numerical results indicate that the CLT interaction removes degeneracies in the spin and p-spin state doublets.

The most stringent interesting result is that the present spin and p-spin symmetric cases can be easily reduced to the KG solution once $S(\boldsymbol{r}) = V(\boldsymbol{r})$ and $S(\boldsymbol{r}) = -V(\boldsymbol{r})$ (i.e., $C_{\rm s} = C_{\rm ps} = 0$).[62] The resulting solutions of the wave functions are expressed in terms of the generalized Jacobi polynomials. Obviously, the relativistic solution can be reduced to that in its non-relativistic limit case by choosing the appropriate mapping transformations.[70] Also, in the case with spin–orbit quantum number $\kappa = 0$, the problem reduces to the s-wave solution.

References

[1] Ginocchio J N 2004 *Phys. Rev. C* **69** 034318
[2] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436
[3] Ginocchio J N 2005 *Phys. Rep.* **414** 165
[4] Page P R, Goldman T and Ginocchio J N 2001 *Phys. Rev. Lett.* **86** 204
[5] Arima A, Harvey M and Shimizu K 1969 *Phys. Lett. B* **30** 517
[6] Hecht K T and Adler A 1969 *Nucl. Phys. A* **137** 129
[7] Ginocchio J N and Madland D G 1998 *Phys. Rev. C* **57** 1167
[8] Bohr A, Hamarnoto I and Motelson B R 1982 *Phys. Scr.* **26** 267
[9] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 *Phys. Rev. Lett.* **59** 1405
[10] Troltenier D, Bahri C and Draayer J P 1995 *Nucl. Phys. A* **586** 53
[11] Meng J, Sugawara-Tanabe K, Yamaji S, Ring P and Arima A 1998 *Phys. Rev. C* **58** R628
[12] Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett. A* **349** 87
[13] Ikhdair S M 2012 *J. Mod. Phys.* **3** 170
[14] Jia C S and de Souza Dutra A 2008 *Ann. Phys.* **323** 566
[15] Ikhdair S M 2009 *Chem. Phys.* **361** 9
[16] Xu Y, He S and Jia C S 2008 *J. Phys. A: Math. Theor.* **41** 255302
[17] Akcay H 2009 *J. Phys. A: Math. Theor.* **42** 198001
[18] de Souza Dutra A and Jia C S 2006 *Phys. Lett. A* **352** 484
[19] Ikhdair S M 2012 *Mol. Phys.* **110** 1415
[20] Sever R and Tezcan C 2008 *Int. J. Mod. Phys. E* **17** 1327
[21] Ikhdair S M and Sever R 2009 *Int. J. Mod. Phys. C* **20** 361
[22] de Souza Dutra A and Almeida C A S 2000 *Phys. Lett. A* **275** 25
[23] Hamzavi M, Ikhdair S M and Solaimani M 2012 *Int. J. Mod. Phys. E* **21** 1250016
[24] Ikhdair S M 2012 *Cent. Eur. J. Phys.* **10** 361
[25] Ikhdair S M and Sever R 2009 *Phys. Scr.* **79** 035002
[26] Ikhdair S M 2009 *Eur. Phys. J. A* **39** 307
[27] Ikhdair S M 2009 *Eur. Phys. J. A* **40** 143
[28] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Berlin: Birkhauser)
[29] Ikhdair S M and Sever R 2011 *J. Math. Phys.* **52** 122108
[30] Boztosun I and Karakoc M 2007 *Chin. Phys. Lett.* **24** 3028
[31] Kocak M 2007 *Chin. Phys. Lett.* **24** 315
[32] Koc R, Tütüncüler H and Olgar E 2006 *Chin. Phys. Lett.* **23** 539
[33] Liang Z, Cao Z Q, Deng X X and Shen Q S 2005 *Chin. Phys. Lett.* **22** 2465
[34] Zhang S Q and Li Z B 2004 *Chin. Phys. Lett.* **21** 223
[35] Hai W H 1998 *Chin. Phys. Lett.* **15** 472
[36] Zhang F L, Yuan B and Liang M L 2007 *Acta Phys. Sin.* **56** 3683 (in Chinese)
[37] Wang Z B and Zhang M C 2006 *Acta Phys. Sin.* **55** 6229 (in Chinese)
[38] Wang Z B and Zhang M C 2006 *Acta Phys. Sin.* **55** 525 (in Chinese)
[39] Wang Z B and Zhang M C 2006 *Acta Phys. Sin.* **55** 521 (in Chinese)
[40] Li N, Ju G X and Ren Z Z 2005 *Acta Phys. Sin.* **54** 2520 (in Chinese)
[41] Chen G 2004 *Acta Phys. Sin.* **53** 684 (in Chinese)
[42] Chen G 2004 *Acta Phys. Sin.* **53** 680 (in Chinese)
[43] Lou Z M and Chen G 2003 *Acta Phys. Sin.* **52** 1075 (in Chinese)
[44] Chen G and Lou Z M 2003 *Acta Phys. Sin.* **52** 1071 (in Chinese)
[45] Chen G 2001 *Acta Phys. Sin.* **50** 1651 (in Chinese)
[46] Chen C Y 2000 *Chin. Phys.* **9** 731
[47] Lu H X, Liu K J, Pan J W and Zhang Y D 2000 *Chin. Phys.* **9** 5
[48] Hou C F, Zhou Z X and Li Y 1999 *Chin. Phys.* **8** 561
[49] Ni Z X 1999 *Chin. Phys.* **8** 8
[50] Chen B Z 1998 *Chin. Phys.* **7** 817
[51] Li L X and Guo G C 1998 *Chin. Phys.* **7** 727
[52] Ni Z X 1998 *Chin. Phys.* **7** 183
[53] Ren T Q, Li D Q, Liu S D and Zhang Q G 1995 *Chin. Phys.* **4** 1
[54] Ikhdair S M and Hamzavi M 2012 *Chin. Phys. B* **21** 110302
[55] Rosen N and Morse P M 1932 *Phys. Rev.* **42** 210
[56] Jia C S, Li S C, Li Y and Sun L T 2002 *Phys. Lett. A* **300** 115
[57] Jia C S, Li Y, Sun Y, Liu J Y and Sun L T 2003 *Phys. Lett. A* **311** 115
[58] Compean C B and Kirchbach M 2006 *J. Phys. A: Math. Gen.* **39** 547
[59] Ma Z Q, Gonzalez-Cisneros A, Xu B W and Dong S H 2007 *Phys. Lett. A* **371** 180
[60] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev. C* **69** 024319
[61] Akçay H 2009 *Phys. Lett. A* **373** 616
[62] Alberto P, Lisboa R, Malheiro M and de Castro A S 2005 *Phys. Rev. C* **71** 034313
[63] Akçay H and Tezcan C 2009 *Int. J. Mod. Phys. C* **20** 930
[64] Mao G 2003 *Phys. Rev. C* **67** 044318
[65] Furnstahl R J, Rusnak J J and Serot B D 1998 *Nucl. Phys. A* **632** 607
[66] Ikhdair S M and Sever R 2010 *Appl. Math. Comput.* **216** 911
[67] Aydoğdu O and Sever R 2010 *Eur. Phys. J. A* **43** 73
[68] Aydoğdu O and Sever R 2011 *Phys. Lett. B* **703** 379
[69] Lu J 2005 *Phys. Scr.* **72** 349
[70] Ikhdair S M and Sever R 2010 *J. Math. Phys.* **51** 023525
[71] Ikhdair S M 2009 *Int. J. Mod. Phys. C* **20** 1563
[72] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
[73] Hamzavi M, Ikhdair S M and Ita B I 2012 *Phys. Scr.* **85** 045009
[74] Ikhdair S M and Hamzavi M 2012 *Phys. Scr.* **86** 045002
[75] Pekeris C L 1934 *Phys. Rev.* **45** 98