International Journal of Modern Physics E
Vol. 22, No. 3 (2013) 1350015 (16 pages)

DOI: 10.1142/S0218301313500158

RELATIVISTIC BOUND STATES IN THE PRESENCE OF SPHERICALLY RING-SHAPED *q*-DEFORMED WOODS–SAXON POTENTIAL WITH ARBITRARY *l*-STATES

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Received 17 October 2012
Revised 7 January 2013
Accepted 23 January 2013
Published 20 March 2013

Approximate bound-state solutions of the Dirac equation with q-deformed Woods–Saxon (WS) plus a new generalized ring-shaped (RS) potential are obtained for any arbitrary l-state. The energy eigenvalue equation and corresponding two-component wave functions are calculated by solving the radial and angular wave equations within a shortcut of the Nikiforov–Uvarov (NU) method. The solutions of the radial and polar angular parts of the wave function are expressed in terms of the Jacobi polynomials. A new approximation being expressed in terms of the potential parameters is carried out to deal with the strong singular centrifugal potential term $l(l+1)r^{-2}$. Under some limitations, we can obtain solution for the RS Hulthén potential and the standard usual spherical WS potential ($q = 1$).

Keywords: Dirac equation; q-deformed Woods–Saxon potential; ring-shaped potential; approximation scheme; Nikiforov–Uvarov method.

PACS Number(s): 03.65.Ge, 03.65.Fd, 0.65.Pm, 02.30.Gp, 21.10.Dr, 21.10.Sf, 24.10.Jv

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1. Introduction

The Woods–Saxon (WS) potential is a realistic short range potential and is used to study the nuclear structure within the shell model in nuclear physics.[1] Different versions of this potential have been introduced to explore elastic and quasi-elastic scattering of nuclear particles.[2] The usual ($q = 1$) and the q-deformed WS potentials have been applied in nuclear calculations,[3–11] studying the behavior of valence electrons in metallic systems or in helium model[12] and in nonlinear scalar theory of mesons.[13]

The Schrödinger equation,[14–18] Klein–Gordon equation,[19–21] Dirac equation[22,23] and Salpeter equation[24] have been solved with the usual and deformed WS potential forms for their bound states in the framework of the Nikiforov–Uvarov method and by other methods.[25–27]

Recently, a new ring-shaped (RS) potential has been introduced plus Coulomb potential,[28] Hulthen potential,[29] modified Kratzer potential[30] and nonharmonic oscillator potential.[31] Such calculations with this RS potential have found applications in quantum chemistry such as the study of RS molecules like benzene. Overmore, the shape form of this potential plays an important role when studying the structure of the deformed nuclei or the nuclear interactions. Quesne[32] obtained a new RS potential by replacing the Coulomb part of the Hartmann potential[33] by a harmonic oscillator term. Gang[34] exactly obtained energy spectrum of some noncentral separable potential in $r$and θ using the method of supersymmetric WKB approximation. Yasuk *et al.*[35] obtained general solutions of Schrödinger equation for a noncentral potential by using the NU method.[36] Yuan *et al.*[37] studied exact solutions of scattering states of the Klein–Gordon equation with Coulomb potential plus new RS potential with equal mixture of scalar and vector potentials. Ikhdair and Sever[38] used the polynomial solution to solve a noncentral potential. Gang and Bang[39] studied the Klein–Gordon with equal scalar and vector Makarov potentials by the factorization method. Kerimov[40] studied nonrelativistic quantum scattering problem for a noncentral potential which belongs to a class of potentials exhibiting an accidental degeneracy. Berkdemir and Sever[41] investigated the diatomic molecules subject to central potential plus RS potential. Also, they[42] solved the pseudospin symmetric solution of the Dirac equation for spin-$\frac{1}{2}$ particles moving with the Kratzer potential connected with an angle-dependent potential systematically. Yeşiltaş[43] showed that a wide class of noncentral potentials can be analyzed via the improved picture of the NU method. Berkdemir and Cheng[44] investigated the problem of relativistic motion of a spin-$\frac{1}{2}$ particle in an exactly solvable potential consisting of harmonic oscillator potential plus a novel RS dependent potential. Zhang *et al.*[45–47] obtained the complete solutions of the Schrödinger and Dirac equations with a spherically harmonic oscillatory RS potential. Ikhdair and Sever obtained the exact solutions of the D-dimensional Schrödinger equation with RS pseudoharmonic potential,[48] modified Kratzer potential[49] and the D-dimensional Klein–Gordon equation with RS pseudoharmonic potential.[50] Hamzavi *et al.* found the exact solutions of the Dirac equation with Hartmann potential[51] and RS pseudoharmonic oscillatory potential[52] by using the NU method. Many authors have also studied a few noncentral potentials within the supersymmetric quantum mechanics and point canonical transformations.[53–55]

Recently, Chabab *et al.*[56] obtained analytical l-state solutions of the Klein–Gordon equation for q-deformed WS plus generalized RS potential for the two cases of equal and different mixed vector and scalar potentials. Very recently, we obtained an approximate bound-state solutions of the Dirac equation with Hulthen potential plus a new generalized RS potential with any arbitrary l-state.[57] The aim of this work is to investigate analytical bound-state solutions of the Dirac equation with noncentral q-deformed WS potential plus a new generalized RS potential with extra additional parameter α from the RS potential used in Ref. 28. Therefore, the noncentral potential of the type $V(\mathbf{r}) = V_{\rm WS}(r) + \frac{1}{r^2}V_{\rm RS}(\theta)$, consisting of two parts

$$V_{\rm WS}(r) = -\frac{V_0}{1+qe^{(r-R_0)/a}}\,, \quad V_{\rm RS}(\theta) = \frac{\alpha+\beta\cos^2\theta}{\sin^2\theta}\,, \tag{1}$$

with $V_{\rm WS}(r)$ is the q-deformed WS potential in which V_0, R_0, a and q are the potential depth, width or nuclear radius, surface thickness and deformation parameters, respectively. Further, $V_{\rm RS}(\theta)$ is a new RS potential identical to the RS part of the Hartmann potential.[28] Here, $\alpha = -p\sigma^2\eta^2 a_0^2\varepsilon_0$ and $\beta = -p\sigma^2\eta^2 a_0^2\varepsilon_0$, where $a_0 = \frac{\hbar^2}{me^2}$ and $\varepsilon_0 = -\frac{me^4}{2\hbar^2}$ represent the Bohr radius and the ground state energy of the hydrogen atom, respectively. Further, η, σ and p are three dimensionless parameters. Generally speaking, η and σ vary from about 1 up to 10 and p is a real parameter and its value is taken as 1.

We also show that when the deformation parameter $q$takes a particular value, the results turn to be the solution for the Hulthen potential.

In our solution, we are using a powerful shortcut of the NU method[36] that has proven its efficiency and easy handling in the treatment of problems with second-order differential equations of the type $y''+(\tilde{\tau}/\sigma)y'+(\tilde{\sigma}/\sigma^2)y = 0$ which are usually encountered in physics such as the radial and angular parts of the Schrödinger, KG and Dirac equations.[48–52]

This paper is organized as follows. In Sec. 2, we present the Dirac equation for the generalized RS q-deformed WS potential. Section 3 is devoted to derive the approximate analytic bound-state energy eigenvalue equation and the associated two-components of the wave function consisting from radial and angular parts within a shortcut of the NU method. Section 4 presents the conclusion of our work.

2. Dirac Equation with Scalar and Vector q-Deformed WS Plus RS Potential

The Dirac equation for a particle of mass M moving in the field of attractive scalar potential $S(\mathbf{r})$ and repulsive vector potential $V(\mathbf{r})$ potentials (in the relativistic units $\hbar = c = 1$) takes the form[58]:

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(M + S(r)) + V(r)]\psi(\mathbf{r}) = E\psi(\mathbf{r})\,, \tag{2}$$

with E as the relativistic energy of the system and $\mathbf{p} = -i\boldsymbol{\nabla}$ as the three-dimensional (3D) momentum operator. Further, $\boldsymbol{\alpha}$ and β represent the 4×4 usual Dirac matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad i = 1, 2, 3 \tag{3}$$

which are expressed in terms of the three 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and I is the 2×2 unitary matrix. In addition, the Dirac wave function $\psi(\mathbf{r})$ can be expressed in Pauli–Dirac representation as

$$\psi(\mathbf{r}) = \begin{pmatrix} \varphi(\mathbf{r}) \\ \chi(\mathbf{r}) \end{pmatrix}. \tag{5}$$

Inserting Eqs. (3)–(5) into Eq. (2) give

$$\boldsymbol{\sigma} \cdot \mathbf{p}\chi(\mathbf{r}) = (E - M - \Sigma(\mathbf{r}))\varphi(\mathbf{r})\,, \tag{6a}$$

$$\boldsymbol{\sigma} \cdot \mathbf{p}\varphi(\mathbf{r}) = (E + M - \Delta(\mathbf{r}))\chi(\mathbf{r})\,, \tag{6b}$$

where the sum and difference potentials, respectively, are defined by

$$\Sigma(\mathbf{r}) = V(\mathbf{r}) + S(\mathbf{r}) \quad \text{and} \quad \Delta(\mathbf{r}) = V(\mathbf{r}) - S(\mathbf{r})\,. \tag{7}$$

For a limiting case when $S(\mathbf{r}) = V(\mathbf{r})$, then $\Sigma(\mathbf{r}) = 2V(\mathbf{r})$ and $\Delta(\mathbf{r}) = 0$. Consequently, Eq. (6) becomes

$$\boldsymbol{\sigma} \cdot \mathbf{p}\chi(\mathbf{r}) = (E - M - 2V(\mathbf{r}))\varphi(\mathbf{r})\,, \tag{8a}$$

$$\chi(\mathbf{r}) = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + M}\varphi(\mathbf{r})\,, \tag{8b}$$

where $E \neq -M$, which means that only the positive energy states exist for a finite lower-component $\chi(\mathbf{r})$ of the wave function.

Combining Eq. (8b) into Eq. (8a) and inserting the potential (1), one can obtain

$$\left[\nabla^2 + E^2 - M^2 + 2(E + M)\left(\frac{V_0}{1 + qe^{(r-R_0)/a}} - \frac{\alpha + \beta\cos^2\theta}{r^2\sin^2\theta}\right)\right]\varphi_{nlm}(r, \theta, \phi) = 0\,, \tag{9}$$

where

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{10}$$

and

$$\varphi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi), \quad R_{nl}(r) = r^{-1}U_{nl}(r), \quad Y_l^m(\theta,\phi) = \Theta_l(\theta)\Phi_m(\phi)\,. \tag{11}$$

After substituting Eqs. (10) and (11) into Eq. (9) and making a separation of variables, we obtain the following sets of second-order differential equations:

$$\frac{d^2U_{nl}(r)}{dr^2} + \left[E^2 - M^2 - \frac{\lambda}{r^2} + \frac{2(E+M)V_0}{1+qe^{(r-R_0)/a}}\right]U_{nl}(r) = 0\,, \tag{12a}$$

$$\frac{d^2\Theta_l(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta_l(\theta)}{d\theta} + \left[\lambda - \frac{m^2}{\sin^2\theta} - \frac{2(E+M)(\alpha+\beta\cos^2\theta)}{\sin^2\theta}\right]\Theta_l(\theta) = 0\,, \tag{12b}$$

$$\frac{d^2\Phi_m(\phi)}{d\varphi^2} + m^2\Phi_m(\phi) = 0\,, \tag{12c}$$

where m^2 and $\lambda = l(l+1)$ are two separation constants with l as the rotational angular momentum quantum number.

The solution of Eq. (12c) is periodic and must satisfy the periodic boundary condition

$$\Phi_m(\phi + 2\pi) = \Phi_m(\phi)\,, \tag{13}$$

which gives the solution:

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\exp(\pm im\phi), \quad m = 0, 1, 2, \ldots. \tag{14}$$

The solutions of the radial part (12a) and polar angular part (12b) equations will be shown in the later section.

3. Analytical Solutions of the Radial and Polar Angle Parts of Dirac Equation

3.1. *Solution of polar angle part*

To obtain the energy eigenvalues and eigenfunctions of the polar angular part of Dirac equation (12b), we make an appropriate transformation of parameter as $z = \cos^2\theta$ (or $z = \sin^2\theta$) to reduce it as

$$\Theta_l''(z) + \left[\frac{(1/2)-(3/2)z}{z(1-z)}\right]\Theta_l'(z) + \frac{1}{z^2(1-z)^2} \times \left[-\frac{1}{4}[\lambda + 2(E+M)\beta]z^2 + [\lambda - m^2 - 2(E+M)\alpha]z\right]\Theta_l(z) = 0\,, \tag{15}$$

where $\Theta_l(z=0) = 0$ and $\Theta_l(z=1) = 0$. The solution of the above angular equation can be easily found by using the shortcut of the NU method presented in Appendix A. Now, in comparing the above equation with Eq. (A.2), we identify the following constants:

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}, \quad c_3 = 1, \quad A = \frac{1}{4}[\lambda + 2(E+M)\beta],$$
$$B = [\lambda - m^2 - 2(E+M)\alpha], \quad C = 0\,.$$

The remaining constants are thus calculated via (A.5) as

$$\begin{aligned}
c_4 &= \frac{1}{4}, \quad c_5 = -\frac{1}{4}, \quad c_6 = \frac{1+4\lambda+8(E+M)\beta}{4}\,,\\
c_7 &= \frac{m^2-\lambda+2(E+M)\alpha}{4} - \frac{1}{8}, \quad c_8 = \frac{1}{16}\,,\\
c_9 &= \frac{m^2+2(E+M)\,(\alpha+\beta)}{4}, \quad c_{10} = \frac{1}{2}\,,\\
c_{11} &= \sqrt{m^2+2(E+M)\,(\alpha+\beta)}, \quad c_{12} = \frac{1}{2}\,,\\
c_{13} &= \frac{1}{2}\sqrt{m^2+2(E+M)\,(\alpha+\beta)}\,.
\end{aligned} \tag{16}$$

We use the energy relation (A.10) and the parametric coefficients given by Eqs. (15) and (16) to obtain a relationship between the separation constant λ and the new nonnegative angular integer $\tilde{n}$ as,

$$\lambda = l(l+1) = \left(2\tilde{n} + \sqrt{m^2+2(E+M)(\alpha+\beta)} + \frac{3}{2}\right)^2 - 2(E+M)\beta - \frac{1}{4}\,, \tag{17a}$$

$$l = \sqrt{\left(2\tilde{n} + \sqrt{m^2+2(E+M)(\alpha+\beta)} + \frac{3}{2}\right)^2 - 2(E+M)\beta} - \frac{1}{2}\,. \tag{17b}$$

Once the RS potential disappears after setting the potential parameters to zero, i.e., $\alpha = \beta = 0$ or simply the angular part $V_{\mathrm{RS}}(\theta) = 0$, we obtain $l = 2\tilde{n} + |m| + 1, m = 0, 1, 2, \ldots$. The angular part of the potential (1), $V_{\mathrm{RS}}(\theta)$, is found to have singularities at angles $\theta = P\pi$ $(P = 0, 1, 2, 3, \ldots)$ as well as at very small and very large values of r.

Let us find the corresponding polar angular part of the wave function. We find the weight function via (A.11) as

$$\rho(z) = z^{1/2}(1-z)^{\sqrt{m^2+2(E+M)(\alpha+\beta)}}\,, \tag{18}$$

which gives the first part of the angular wave function via (A.13) in terms of the Jacobi polynomial as

$$y_{\tilde{n}}(z) \sim P_{\tilde{n}}^{(1/2,\sqrt{m^2+2(E+M)(\alpha+\beta)})}(1-2z)\,. \tag{19}$$

The second part of the angular wave function can be obtained via (A.12) as

$$\varphi(z) \sim z^{1/2}(1-z)^{\sqrt{m^2+2(E+M)(\alpha+\beta)}/2} \tag{20}$$

and, hence, the angular part of the wave function can be obtained via Eq. (A.14); namely, $\Theta_l(z) = \varphi(z)y_{\tilde{n}}(z)$ as

$$\Theta_l(\theta) = A_{\tilde{n}}\cos\theta(\sin\theta)^{\sqrt{m^2+2(E+M)(\alpha+\beta)}}P_{\tilde{n}}^{(1/2,\sqrt{m^2+2(E+M)(\alpha+\beta)})}(1-2\cos^2\theta)\,, \tag{21}$$

where $A_{\tilde{n}}$ is the normalization factor. When the RS potential disappears, i.e., $\alpha = \beta = 0$, then

$$\Theta_l(\theta) = A_{\tilde{n}}\cos\theta\sin^{|m|}\theta P_{\tilde{n}}^{(1/2,|m|)}(1-2\cos^2\theta)\,.$$

3.2. *Solution of radial part equation*

In this part, we will consider the energy eigenvalue equation and the wave function of the radial part of the Dirac equation with the q-deformed WS potential. The exact solution is not handy due to existence of the strong singular centrifugal potential term λr^{-2} in Eq. (12a). Therefore, an approximate analytical solution has been done for this term in Appendix B by using the following change of variable, $x = (r-R_0)/R_0$ and $\gamma = R_0/a$. Thus, Eq. (12a) becomes

$$U''_{nl}(x) + [R_0^2(E^2-M^2) - \lambda(D_0 + D_1 v + D_2 v^2) + 2(E+M)V_0R_0^2 v]U_{nl}(x) = 0\,,$$
$$v = \frac{1}{1+qe^{\gamma x}}\,, \tag{22}$$

where the explicit forms of the constants D_i $(i = 0,1,2)$ are derived explicitly in Appendix B. Furthermore, making a change of variables as $s = e^{\gamma x}$, we can recast Eq. (22) into the simple form

$$U''_{nl}(s) + \frac{(1+qs)}{s(1+qs)}U'_{nl}(s) + \frac{[-\tilde{E}q^2s^2 + (\delta-2\tilde{E})qs - (\tilde{E}+\eta-\delta)]}{s^2(1+qs)^2}U_{nl}(s) = 0\,, \tag{23}$$

with

$$\tilde{E} = \frac{\lambda D_0 - R_0^2(E^2-M^2)}{\gamma^2}, \quad \delta = \frac{2(E+M)V_0R_0^2 - \lambda D_1}{\gamma^2}, \quad \eta = \frac{\lambda D_2}{\gamma^2}\,. \tag{24}$$

Comparing Eq. (23) with its counterpart hypergeometric equation (A.2), we identify values of the following constants:

$$c_1 = 1, \quad c_2 = -q, \quad c_3 = -q, \quad A = q^2\tilde{E}, \quad B = q(\delta - 2\tilde{E}), \quad C = \tilde{E} + \eta - \delta \tag{25}$$

and the remaining constants are calculated via (A.5) as

$$c_4 = 0, \quad c_5 = \frac{q}{2}, \quad c_6 = \frac{q^2}{4}(1+4\tilde{E}), \quad c_7 = q(2\tilde{E}-\delta), \quad c_8 = \tilde{E}+\eta-\delta\,,$$
$$c_9 = \frac{q^2}{4}(1+4\eta), \quad c_{10} = 2\sqrt{\tilde{E}+\eta-\delta}, \quad c_{11} = -\frac{1}{q}\sqrt{q^2(1+4\eta)}\,, \tag{26}$$
$$c_{12} = \sqrt{\tilde{E}+\eta-\delta}, \quad c_{13} = \frac{1}{2}\left(1 - \frac{1}{q}\sqrt{q^2(1+4\eta)}\right)\,.$$

The energy eigenvalue equation can be obtained via the relation (A.10) and values of constants given by Eq. (25) and Eq. (26) after lengthy but straightforward algebra as

$$E^2 - M^2 + 2(E+M)V_0 = \frac{l(l+1)}{R_0^2}(D_0 + D_1 + D_2) - \frac{a^2}{4}\left[\frac{2(E+M)V_0 - l(l+1)(D_1+D_2)/R_0^2}{n + \frac{1}{2} - \frac{1}{2q}\sqrt{q^2\left(1 + \frac{4l(l+1)D_2}{a^2R_0^2}\right)}} - \frac{1}{a^2}\left(n + \frac{1}{2} - \frac{1}{2q}\sqrt{q^2\left(1 + \frac{4l(l+1)D_2}{a^2R_0^2}\right)}\right)\right]^2, \quad (27)$$

and recalling that $l(l+1) = (2\tilde{n} + \sqrt{m^2 + 2(E+M)(\alpha+\beta)} + \frac{3}{2})^2 - 2(E+M)\beta - \frac{1}{4}$.

On the other hand, in the nonrelativistic limiting case where $E + M \simeq 2M$ and $E - M \simeq E$, Eq. (27) becomes

$$E_{nl'm} = -V_0 + \frac{l'(l'+1)}{2MR_0^2}(D_0 + D_1 + D_2) - \frac{a^2}{8M}\left[\frac{4MV_0 - l'(l'+1)(D_1+D_2)/R_0^2}{n + \frac{1}{2} - \frac{|q|}{2q}\sqrt{1 + \frac{4l'(l'+1)D_2}{a^2R_0^2}}} - \frac{1}{a^2}\left(n + \frac{1}{2} - \frac{|q|}{2q}\sqrt{1 + \frac{4l'(l'+1)D_2}{a^2R_0^2}}\right)\right]^2, \quad (28)$$

where $l'(l'+1) = (2\tilde{n} + \sqrt{m^2 + 4M(\alpha+\beta)} + \frac{3}{2})^2 - 4M\beta - \frac{1}{4}$.

For numerical solution of the energy equations (27) and (28) with parameter values $M = 10$ fm^{-1}, $R_0 = 7$ fm, $a = 0.5$ fm and $V_0 = 5$ fm^{-1} (Ref. 59), we approximately calculate the energy eigenvalues for the usual WS potential ($q = 1$) plus the RS potential ($\alpha = 1$ and $\beta = 1$) and compare when the RS potential is zero ($\alpha = 0$ and $\beta = 0$). The results are shown in Table 1.

Figures 1–3 show the behavior of energy of the Dirac equation with WS potential plus RS potential versus the potential parameters: surface thickness a, the deformation parameter q and nuclear radius R_0, respectively, for various values n, $\tilde{n}$ and m. Figure 1 shows that, as the value of a increases, the energy value for which

Table 1. The bound-state energy eigenvalues of the usual WS potential plus RS potential for various values of n, $\tilde{n}$ and m quantum numbers.

			$\alpha=\beta=1$		$\alpha=\beta=0$	
n	$\tilde{n}$	m	$E_{n,\tilde{n},m}$ Relativistic	$E_{n,\tilde{n},m}$ Nonrelativistic	$E_{n,\tilde{n},m}$ Relativistic	$E_{n,\tilde{n},m}$ Nonrelativistic
1	0	1	−8.181582591	−15.72190029	−8.087401684	−13.64437737
1	0	1	−8.199178439	−16.35979304	−8.245532350	−15.00275622
1	1	0	−8.390837180	−23.88322236	−8.449864847	−17.25283431
1	1	1	−8.930386266	−25.42691358	−8.676477943	−20.76166288
2	0	0	−5.288469326	−5.163723495	−4.511739044	−5.131551195
2	0	1	−5.32440643	−5.175961484	−4.671607147	−5.32440643
2	1	0	−5.923074363	−5.351169122	−4.890413294	−5.194344782
2	1	1	−5.962042737	−5.388986308	−5.153319654	−5.274965974
2	2	0	−6.553753850	−5.829767768	−5.448307237	−5.409155814
2	2	1	−6.595079988	−5.915307507	−5.765975978	−5.617085700
3	0	0	−2.480509032	−5.593696192	−1.844569412	−5.809477801
3	0	1	−2.501689705	−5.537243884	−1.938721382	−5.662367950
3	1	0	−2.967053503	−5.073595802	−2.069933874	−5.464359829
3	1	1	−2.990765343	−5.007785930	−2.231472167	−5.231175524

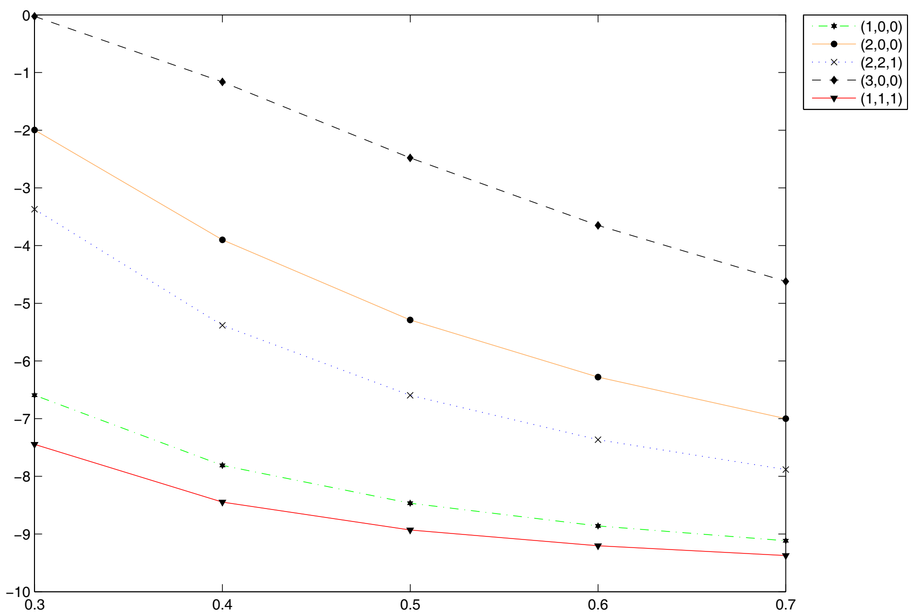

Fig. 1. Energy behavior of the Dirac equation with WS potential plus RS potential versus the diffuseness of the nuclear surface a for various n, $\tilde{n}$ and m, respectively.

antiparticle states appear decreases; they become more negative i.e., the particle becomes more attractive by the potential (1). In Figs. 2 and 3, we observe that, as the values of q and R_0 increase, respectively, the energy value for which antiparticle states appear increases, for the selected range, in the positive direction (becomes

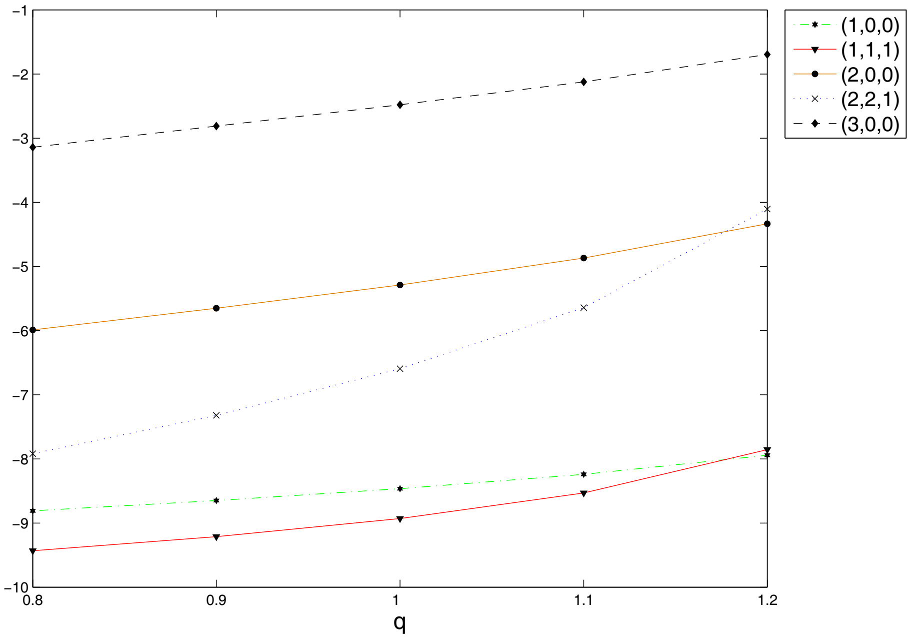

Fig. 2. Energy behavior of the Dirac equation with WS potential plus RS potential versus deformation parameter q for various n, $\tilde{n}$ and m.

less negative) for any given state $(n, \tilde{n}, m)$, i.e., the particle becomes less attractive to potential (1).

Now, we turn to calculate the corresponding radial part of the wave function. The first step, we find the weight function via (A.11) as

$$\rho(s) = s^{2p_0}(1+qs)^{w_0}\,, \tag{29}$$

where

$$p_0 = \frac{1}{\gamma}\sqrt{R_0^2[M^2 - E^2 - 2(E+M)V_0] + l(l+1)(D_0 + D_1 + D_2)} > 0\,, \tag{30a}$$

$$w_0 = -\frac{1}{q}\sqrt{q^2\left(1 + \frac{4l(l+1)D_2}{\gamma^2}\right)}\,. \tag{30b}$$

Hence, using Eq. (29), the first part of the radial wave function can be obtained by means of the relation (A.13) in terms of the Jacobi polynomials as,

$$y_n(s) \sim P_n^{(2p_0, w_0)}(1+2qs)\,. \tag{31}$$

The second part of the radial wave function can be obtained via (A.12) as

$$\phi(s) \sim s^{p_0}(1+qs)^{(1+w_0)/2} \tag{32}$$

and, hence, the radial part of the wave function, $U_{nl}(s) = \phi(s)y_n(s)$ is

$$U_{nl}(r) = B_{nl}(e^{(r-R_0)/a})^{p_0}(1+qe^{(r-R_0)/a})^{(1+w_0)/2}P_n^{(2p_0,w_0)}(1+2qe^{(r-R_0)/a})\,, \tag{33}$$

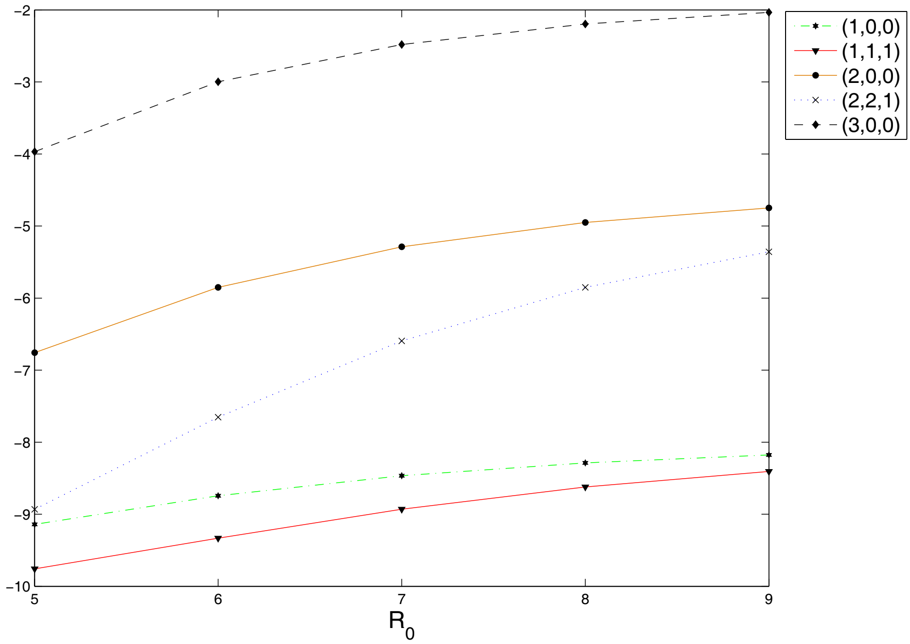

Fig. 3. Energy behavior of the Dirac equation with WS potential plus RS potential versus width of the potential R_0 for various n, $\tilde{n}$ and m, respectively.

where B_{nl} is the normalization constant and we have used the definition of the Jacobi polynomials given by,[60]

$$P_n^{(a,b)}(y) = \frac{(-1)^n}{n!2^2}(1-y)^{-a}(1+y)^{-b}\frac{d^n}{dy^n}[(1-y)^{a+n}(1+y)^{b+n}] .$$

To compute the normalization constant B_{nl}, it is easy to show with the use of $R_{nl}(r) = r^{-1}U_{nl}(r)$, that

$$\int_0^\infty |R_{nl}(r)|^2 r^2 dr = \int_0^\infty |U_{nl}(r)|^2 dr = \int_0^\infty |U_{nl}(s)|^2 \frac{ads}{s} = 1 , \quad (34)$$

where we have used the substitution $s = e^{(r-R_0)/a}$. In our case, with the aid of (A.15), the Jacobi polynomials can be expressed in terms of the hypergeometric function as[60]

$$P_n^{(2p_0,w_0)}(1+2qe^{(r-R_0)/a}) = \frac{\Gamma(n+2p_0+1)}{n!\Gamma(2p_0+1)} {}_2F_1(-n, 2p_0+w_0+n+1; 1+2p_0; qe^{(r-R_0)/a}) . \quad (35)$$

Finally, combining Eqs. (14), (21) and (33), the total upper-component of the wave function (11) becomes

$$\varphi(\mathbf{r}) = \varphi_{nlm}(r,\theta,\phi) = N_{nlm}\frac{1}{\sqrt{2\pi}}\cos\theta(\sin\theta)^{\sqrt{m^2+2(E+M)(\alpha+\beta)}}$$

$$\times P_{\tilde{n}}^{(1/2,\sqrt{m^2+2(E+M)(\alpha+\beta)})}(1-2\cos^2\theta)e^{\pm im\phi}(e^{(r-R_0)/a})^{p_0}$$
$$\times (1+qe^{(r-R_0)/a})^{(1+w_0)/2}P_n^{(2p_0,w_0)}(1+2qe^{(r-R_0)/a})\,, \tag{36}$$

where $m = 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, and recalling that

$$l = \sqrt{\left(2\tilde{n} + \sqrt{m^2+2(E+M)(\alpha+\beta)} + \frac{3}{2}\right)^2 - 2(E+M)\beta} - \frac{1}{2}, \quad l = 0, 1, 2, \ldots .$$

The lower-component of the wave function (5) can be found by means of Eq. (8b) as

$$\chi(\mathbf{r}) = \chi_{nlm}(r,\theta,\phi) = N_{nlm}\frac{1}{\sqrt{2\pi}}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+M}\cos\theta(\sin\theta)^{\sqrt{m^2+2(E+M)(\alpha+\beta)}}$$
$$\times P_{\tilde{n}}^{(1/2,\sqrt{m^2+2(E+M)(\alpha+\beta)})}(1-2\cos^2\theta)e^{\pm im\phi}(e^{(r-R_0)/a})^{p_0}$$
$$\times (1+qe^{(r-R_0)/a})^{(1+w_0)/2}P_n^{(2p_0,w_0)}(1+2qe^{(r-R_0)/a}), \quad E \neq -M\,. \tag{37}$$

For the case of the RS Hulthen potential, we make the following simple changes: $q = -e^{R_0/a}$ and $V_0 = -V_0'$ in the expressions (27), (30), (33), (36) and (37).

4. Final Remarks and Conclusion

In this work, we have investigated the approximate bound state solutions of the Dirac equation with the q-deformed WS plus a new RS potential for any orbital l quantum numbers. By making an appropriate approximation to deal with the centrifugal potential term, we have obtained the energy eigenvalue equation and the unnormalized two spinor components of the wave function $\varphi(\mathbf{r})$ and $\chi(\mathbf{r})$ expressed in terms of the Jacobi polynomials. This problem is solved within the shortcut of the NU method introduced recently in Ref. 61. The relativistic solution can be reduced into the Schrödinger solution under the nonrelativistic limit, to the Hulthen solution and to RS usual WS potential with ($q = 1$).

Acknowledgments

The authors wish to thank the referees for their invaluable suggestions and critics which have greatly helped in improving this paper.

Appendix A. A Shortcut of the NU Method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation $s = s(r)$,[36]

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{A.1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree and $\tilde{\tau}(s)$ is a first degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution, we present a shortcut for the method. So, at first, we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows[61–65]:

$$\psi_n''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi_n'(s) + \frac{(-As^2 + Bs - C)}{s^2(1 - c_3 s)^2}\psi_n(s) = 0\,, \tag{A.2}$$

satisfying the wave functions

$$\psi_n(s) = \phi(s) y_n(s)\,. \tag{A.3}$$

Comparing (A.2) with its counterpart (A.1), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -As^2 + Bs - C\,. \tag{A.4}$$

Following the NU method,[36] we obtain the following necessary parameters[61]:

(i) Relevant constant:

$$\begin{gathered} c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3)\,, \\ c_6 = c_5^2 + A, \quad c_7 = 2c_4 c_5 - B\,, \\ c_8 = c_4^2 + C, \quad c_9 = c_3(c_7 + c_3 c_8) + c_6\,, \\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1\,, \\ c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0\,, \\ c_{12} = c_4 + \sqrt{c_8} > 0, \quad c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_3} - c_5) > 0, \quad c_3 \neq 0\,. \end{gathered} \tag{A.5}$$

(ii) Essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]\,, \tag{A.6}$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}\,, \tag{A.7}$$

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]\,, \tag{A.8}$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0\,. \tag{A.9}$$

(iii) Energy equation:

$$\begin{aligned} &c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 \\ &\quad + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0\,. \end{aligned} \tag{A.10}$$

(iv) Wave functions:

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}}\,, \tag{A.11}$$

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0\,, \tag{A.12}$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s), \quad c_{10} > -1, \quad c_{11} > -1\,, \tag{A.13}$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 s) \tag{A.14}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$ and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(a_0,b_0)}(1 - 2s) = \frac{(a_0 + 1)_n}{n!} {}_2F_1(-n, 1 + a_0 + b_0 + n; a_0 + 1; s) \tag{A.15}$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}} (1 - c_3 s)^{c_{13}} {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 s)\,, \tag{A.16}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

Appendix B. Approximation to the Strong Singular Orbital Centrifugal Term

Here, we make a new approximation to deal with the strong singular centrifugal potential given in Eq. (12a). The centrifugal term is expanded around $r = R_0$ or $x = 0$ in a series of powers of $x = (r - R_0)/(r - R_0)/R_0 \in (-1, \infty)$ and $\gamma = R_0/a$, as

$$V_l(r) = \frac{\lambda}{r^2} = \frac{\lambda}{R_0^2(1 + x)^2} = \frac{\lambda}{R_0^2}(1 - 2x + 3x^2 - \cdots), \quad x \ll 1\,, \tag{B.1}$$

where $\lambda = l(l + 1)$. The above centrifugal potential (B.1) can be replaced by the form formally homogeneous to the original q-deformed WS potential to keep the factorization of the corresponding Schrödinger-like equation. Thus, we take the centrifugal potential in the form

$$\tilde{V}_l(r) = \frac{\lambda}{R_0^2}\left[D_0 + D_1 \frac{1}{1 + qe^{\gamma x}} + D_2 \frac{1}{(1 + qe^{\gamma x})^2}\right], \quad x \ll 1/\gamma, \quad \gamma x \ll 1\,, \tag{B.2}$$

where $\gamma = R_0/a$ and D_i are the parameter of coefficients ($i = 0, 1, 2$). After making a Taylor expansion to (B.2) up to the second-order term, x^2, and then comparing equal powers with (B.1), we can readily determine D_i ($i = 0, 1, 2$) parameters of the generalized WS potential as function of the specific potential parameters R_0, γ and q as follows:

$$D_0 = 1 - \left(3 + \frac{2}{q} - \frac{1}{q^2}\right)\frac{1}{\gamma} + 3\left(1 + \frac{2}{q} + \frac{1}{q^2}\right)\frac{1}{\gamma^2}\,, \tag{B.3}$$

$$D_1 = 2\left(3 + 2q - \frac{1}{q^2}\right)\frac{1}{\gamma} - 6\left(3 + q + \frac{3}{q} + \frac{1}{q^2}\right)\frac{1}{\gamma^2} \tag{B.4}$$

and

$$D_2 = -\left(2q + 3q^2 - \frac{2}{q} - \frac{1}{q^2}\right)\frac{1}{\gamma} + 3\left(6 + 4q + q^2 + \frac{4}{q} + \frac{1}{q^2}\right)\frac{1}{\gamma^2}\,. \tag{B.5}$$

When $q = 1$, we obtain an approximation for the centrifugal potential in case of usual WS potential as

$$D_0 = 4\left(\frac{3}{\gamma^2} - \frac{1}{\gamma}\right) + 1, \quad D_1 = 8\left(\frac{1}{\gamma} - \frac{6}{\gamma^2}\right), \quad D_2 = 2\left(\frac{24}{\gamma^2} - \frac{1}{\gamma}\right). \tag{B.6}$$

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