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APPLIED MATHEMATICS AND COMPUTATION
ELSEVIER Applied Mathematics and Computation 159 (2004) 797–807
www.elsevier.com/locate/amc

Preconditioned conjugate gradient method for three-dimensional non-convex enclosure geometries with diffuse and grey surfaces

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Abstract

Our main concern in this paper is the numerical simulation of the heat radiation exchange in a three-dimensional non-convex enclosure geometry with a diffuse and grey surface. This physical phenomena is governed by a boundary integral equation of the second kind. Due to the non-convexity of the enclosure the presence of the shadow zones must be taken into account in the heat radiation analysis. For that purpose we have developed a geometrical algorithm to provide an efficient detection of these shadow zones that are needed to calculate the visibility function. For the discretization of the boundary integral equation we have used the boundary element method based on the Galerkin–Bubnov scheme. The system of linear equations which subsequently arise has been solved by the conjugate gradient method with preconditioning. To demonstrate the high efficiency of this method a numerical experiment has been constructed for non-convex geometry; the heat radiation in an aperture has been considered.

Keywords: Heat radiation; Non-convex geometries; Fredholm integral equations; Galerkin scheme; Conjugate gradient method; Boundary element method

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doi:10.1016/j.amc.2003.11.009

1. Introduction

All substances continuously emit electromagnetic radiation by virtue of the molecular and atomic agitation associated with the internal energy of the material. In the equilibrium state, this internal energy is in direct proportion to the temperature of the substance. The emitted radiant energy can range from radio waves, which can have wavelength of miles, to cosmic rays with wavelength of $>10^{-14}$ m. Heat radiation plays a very significant role in our modern technology. It has to be taken into account if the temperature on a visible surface of the system is high enough or if other heat transfer mechanisms are not present (like in a vacuum, for example). Apart from some simple cases like a convex radiation body and known irradiation from infinity, we have to take into account the radiative heat exchange between different parts of the surface of our system. The heat radiative exchange phenomena is governed by a boundary integral equation. Concerning this integral equation, we are aware of some independent works: In [5,6] a boundary element method was implemented for two-dimensional enclosures to obtain a direct numerical solution for the integral equation, however, this leads to quite high error bounds. In [11,12] two-dimensional convex and non-convex geometries have been considered and some solution methods for the discrete heat equation have been compared. Some higher-order numerical methods for solving the radiosity equation in the convex case are defined and analysed in [2,3].

Our main emphasis in this current work is the radiosity equation defined for a three-dimensional non-convex enclosure geometry with diffuse and grey surfaces. In such geometries the presence of the shadow zones must be taken into considerations due to their direct contribution to the calculation of the visibility function. For the numerical simulation of the integral equation we use the boundary element method based on the Galerkin discretization scheme. Consequently, the conjugate gradient algorithm with preconditioning has been implemented for a three-dimensional non-convex geometry (we take an aperture for example) to calculate the outgoing flux. This method has proved to be very efficient for this case.

2. The formulation of the problem

We consider a three-dimensional non-convex enclosure with a diffuse and grey surface Γ as shown in Fig. 1. We say that a surface is diffuse as emitter (reflector) if it emits (reflects) heat uniformly in every direction. For a grey surface the emissivity and reflectivity are independent of the wavelength (color) of the radiation. Thus, only the total intensity of radiation and not its spectrum is needed in a heat balance model. On Γ we assume, for simplicity, that the temperature is given.

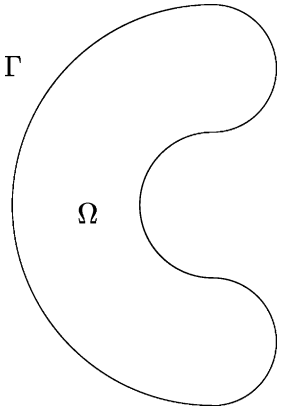

Fig. 1. Enclosure geometry.

Under the above assumptions the heat balance on Γ is given by

$$R - q_0 + J = 0, \tag{2.1}$$

where R is the heat brought to the surface by conduction, q_0 is the radiation emitted by the surface Γ, J is the energy of incoming irradiation on Γ.

For diffuse and grey surfaces the intensity of emitted radiation has the form

$$q_0 = \varepsilon\sigma T^4 + \varrho J. \tag{2.2}$$

The first part of Eq. (2.2) corresponds to the Stefan–Boltzmann radiation law, with ε is the emissivity coefficient ($0 < \varepsilon < 1$), σ is the Stefan–Boltzmann constant which has the value $5.669996 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}}$, ϱ is the reflection coefficient with the relation $\varrho = 1 - \varepsilon$ for opaque grey surfaces.

Eq. (2.2) with $\varrho = 1 - \varepsilon$ can be rewritten as

$$q_0 = \varepsilon\sigma T^4 + (1 - \varepsilon)J. \tag{2.3}$$

In the case of non-convex geometries, the irradiation on Γ depends only on the radiation emitted by different parts of Γ itself. Consequently, we can write for any point $x \in \Gamma$

$$J(x) = \int_\Gamma W^*(x,y)V(x,y)\,\mathrm{d}\Gamma_y, \tag{2.4}$$

where $W^*(x,y)$ is called the view factor between the points x and y of Γ. For general enclosure geometries $W^*(x,y)$ has the representation [16]

$$W^*(x,y) = \frac{\cos\theta_x \cdot \cos\theta_y}{\pi|x-y|^2} \tag{2.5}$$

or equivalently

$$W^*(x,y) = \frac{[n_y \cdot (y-x)] \cdot [n_x \cdot (x-y)]}{\pi |x-y|^4}. \tag{2.6}$$

In this, n_x is the inner normal to Γ at the point x and θ_x denotes the angle between n_x and $x-y$, n_y and θ_y are defined analogously. The function $V(x,y)$ in (2.4) is called the visibility function. More precisely, if the points x and y can "see each other" along a straight line segment that does not intersect Γ at any other point, then $V(x,y) = 1$; otherwise $V(x,y) = 0$.

Substituting Eq. (2.4) into Eq. (2.3) we obtain for any point $x \in \Gamma$

$$q_0(x) = \varepsilon(x)\sigma T^4(x) + (1-\varepsilon(x)) \int_\Gamma W(x,y) q_0(y)\, \mathrm{d}\Gamma_y \quad \text{for } x \in \Gamma, \tag{2.7}$$

where the kernel $W(x,y)$ is given by

$$W(x,y) = W^*(x,y)V(x,y). \tag{2.8}$$

Since $V(x,y)$ and $W^*(x,y)$ are symmetric functions, it follows that $W(x,y)$ is also symmetric.

Eq. (2.7) is a Fredholm boundary integral equation of the second kind. We introduce the integral operator $\widetilde{K} : L^\infty(\Gamma) \to L^\infty(\Gamma)$ with

$$\widetilde{K} q_0(x) := \int_\Gamma W(x,y) q_0(y)\, \mathrm{d}\Gamma_y \quad \text{for } x \in \Gamma,\ q_0 \in L^\infty(\Gamma). \tag{2.9}$$

The properties of the integral operator (2.9) have been thoroughly investigated in [11,13,17]. We have shown that in the case of a smooth surface Γ the kernel of the integral equation is weakly singular of type $|x-y|^{-2(1-\delta)}$ with $\delta \in [0,1)$ and hence the kernel is integrable. Furthermore, the mapping $\widetilde{K} : L^p(\Gamma) \to L^p(\Gamma)$ is compact for $1 \leqslant p \leqslant \infty$ with $\|\widetilde{K}\| = 1$ in $L^p(\Gamma)$ and for the spectral radius we get $\varrho(\widetilde{K}) = 1$. The application of Banach's fixed point theorem leads to the proof of the existence and uniqueness of the solution $q_0 \in L^p(\Gamma)$ of the radiosity equation (2.7).

3. Numerical treatment of the radiosity integral equation in 3D

3.1. Boundary element method and Galerkin discretization

For the application of the boundary element method in 3D, let $\tau_n = \{\Delta_k\}_{k=1}^n$ denotes a sequence of triangulations of the surface Γ for some sequences of integers $n \to \infty$ and let h be the mesh size. We choose an integer $r \geqslant 0$ and with χ_n we denote the set of all functions ϕ which are piecewise polynomials of degree $\leqslant r$ in the parametrization variables as discussed in [1]. There are no

continuity restrictions to the functions in χ_n. The dimension of χ_n is $n \cdot f_r$, where n is the number of elements in τ_n and

$$f_r := \frac{1}{2}(r+1)(r+2)$$

is the number of independent polynomials of degree $\leqslant r$. In the language of Babŭska and Aziz [4], the approximating space χ_n is an $(r+1,0)$-regular system. The following properties of χ_n can be proved, for details see [4]:

Approximation property: Let $t \leqslant s \leqslant r+1$, $t \leqslant 0$. Then for any $\phi \in H^s(\Gamma)$ and any n, there is an element $\phi_n \in \chi_n$ with

$$\|\phi - \phi_n\|_{H^t(\Gamma)} \leqslant c(s,t)h^{s-l}\|\phi\|_{H^s(\Gamma)}, \tag{3.1}$$

where $c(s,t)$ is independent of ϕ, ϕ_n and n.

Inverse property: Let $s \leqslant t \leqslant 0$. Then for any $\eta \in \chi_n$ it holds

$$\|\eta\|_{H^t(\Gamma)} \leqslant c(s,t)h^{s-l}\|\eta\|_{H^s(\Gamma)} \tag{3.2}$$

with $c(s,t)$ is independent of n.

To this end, the weak formulation of the integral equation (2.7) in $L^2(\Gamma)$ reads: Find $q_0 \in L^2(\Gamma)$ such that for all $u \in L^2(\Gamma)$ there holds

$$\begin{aligned}\int_\Gamma q_0(x)u(x)\,\mathrm{d}\Gamma_x = \sigma \int_\Gamma \varepsilon(x)T^4(x)u(x)\,\mathrm{d}\Gamma_x \\ + \int_\Gamma (1-\varepsilon(x)) \int_\Gamma W(x,y)q_0(y)\,\mathrm{d}\Gamma_y u(x)\,\mathrm{d}\Gamma_x.\end{aligned} \tag{3.3}$$

We consider a Galerkin formulation and choose bilinear trial and basis functions ϕ_k, $k = 1,\ldots,n$ with local support $\Gamma_k \subset \Gamma$. Then the Galerkin equations read:

Find $q_{0,n}(x) = \sum_{i=1}^n q_0^{(i)}\phi_i(x) \in \chi_n$ such that

$$\begin{aligned}\sum_{i=1}^n q_0^{(i)} \int_{\Gamma_j} \phi_i(x)\phi_j(x)\,\mathrm{d}\Gamma_x - \sum_{i=1}^n q_0^{(i)} \int_{\Gamma_j} (1-\varepsilon(x)) \int_{\Gamma_i} W(x,y)\phi_j(x)\phi_i(y)\,\mathrm{d}\Gamma_y\,\mathrm{d}\Gamma_x \\ = \sigma \int_{\Gamma_j} \varepsilon(x)T^4(x)\phi_j(x)\,\mathrm{d}\Gamma_x\end{aligned} \tag{3.4}$$

holds for all $j = 1,2,\ldots,n$.

Eq. (3.4) can be written in the following short form:

$$C_n q_{0,n} := (M_n - S_n)q_{0,n} = f_n, \tag{3.5}$$

using the abbreviations $M = (M_{ij})_{i,j=1,\ldots,n}$ for the mass matrix, $S = (S_{ij})_{i,j=1,\ldots,n}$ for the stiffness matrix and $f = (f_j)_{j=1,\ldots,n}$ for the right-hand side of the discretized equation.

3.2. Properties of the mass and the stiffness matrices

The mass matrix M in (3.5) is symmetric, positive definite and diagonal dominant, hence it is invertible. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of the matrix M respectively. Then the following well known estimates hold:

$$\lambda_{\min}\|q_0\|_{l^2}^2 \leqslant (Mq_0, q_0) \leqslant \lambda_{\max}\|q_0\|_{l^2}^2, \tag{3.6}$$

$$\frac{1}{\lambda_{\max}}\|q_0\|_{l^2}^2 \leqslant (M^{-1}q_0, q_0) \leqslant \frac{1}{\lambda_{\min}}\|q_0\|_{l^2}^2, \tag{3.7}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product of $\mathbb{R}^n$ with $(q_0, q_0) = \|q_0\|_{l^2}^2$. Furthermore, it holds

$$\|M\|_{l^2} = \lambda_{\max}, \quad \frac{1}{\|M^{-1}\|_{l^2}} = \lambda_{\min}.$$

Obviously the system of equations $(M_n - S_n)$ is also symmetric and positive definite.

3.3. The conjugate gradient method

Since the system $C_n = M_n - S_n$ is symmetric and positive definite, the conjugate gradient method (cg-method) can be applied to solve the linear system of equations (3.5). The cg-method is a very effective scheme for solving symmetric and positive definite systems. It is given by the following algorithm, see [7].

1. Choose an initial vector $q_{0,n}^{(0)}$ and compute $r_0 = C_n q_{0,n}^{(0)} - f_n$, set $p_0 = r_0$.
2. For $k \geqslant 0$ compute

$$\alpha_k = \frac{r_k^T, p_k}{p_k^T C_n p_k},$$
$$q_{0,n}^{(k+1)} = q_{0,n}^{(k)} + \alpha_k p_k,$$
$$r_{k+1} = C_n q_{0,n}^{(k+1)}.$$

3. Stop the calculation if

$$\frac{\|r_{k+1}\|_2}{\|r_k\|_2} < \varepsilon.$$

4. Otherwise compute

$$\beta_k = \frac{r_{k+1}^T C_n p_k}{p_k^T C_n p_k},$$
$$p_{k+1} = r_{k+1} + \beta_k p_k.$$

3.4. Convergence of the conjugate gradient method

The convergence analysis is based on the following nice property of the integral operator K, defined as

$$K = (1-\varepsilon)\widetilde{K},$$

where $\widetilde{K}$ is given by Eq. (2.9).

Lemma 1. *The integral operator $A := (I-K)$ is L^2-elliptic. Furthermore A is a positive definite operator which satisfies the Gårding inequality on Γ.*

Proof. From Lemma 3 in [13] it follows that

$$\|Kq_0\|_{L^2(\Gamma)} \leqslant (1-\varepsilon)\|q_0\|_{L^2(\Gamma)}. \tag{3.8}$$

Furthermore, K satisfies the inequality

$$\langle Kq_0, q_0\rangle_{L^2(\Gamma)} \leqslant (1-\varepsilon)\langle q_0, q_0\rangle_{L^2(\Gamma)}. \tag{3.9}$$

Eq. (3.9), together with the definition $A := I - K$ leads to

$$\varepsilon\langle q_0, q_0\rangle_{L^2(\Gamma)} \leqslant \langle Aq_0, q_0\rangle_{L^2(\Gamma)} \leqslant (2-\varepsilon)\langle q_0, q_0\rangle_{L^2(\Gamma)}. \tag{3.10}$$

Moreover, A satisfies the Gårding inequality, i.e., for all $q_0 \in L^2(\Gamma)$ and $\varepsilon \geqslant 0$ the following holds

$$\mathrm{Re}\langle Aq_0, q_0\rangle_{L^2(\Gamma)} = \mathrm{Re}\int_\Gamma q_0 A q_0 \,\mathrm{d}\Gamma_x \geqslant \varepsilon\|q_0\|^2_{L^2(\Gamma)}. \qquad \square \tag{3.11}$$

We now let $q_0 \in \chi_n \subset L^2(\Gamma)$ and then define

$$q_0(t) = \sum_{i=1}^{n} q_0^{(i)}\phi_i(t). \tag{3.12}$$

Substituting Eq. (3.12) into Eq. (3.10) leads to

$$\begin{aligned}\varepsilon\left\|\sum_{i=1}^{n} q_0^{(i)}\phi_i\right\|^2_{L^2(\Gamma)} &\leqslant \sum_{i,j=1}^{n} q_0^{(i)} q_0^{(i)}\langle A\phi_i, \phi_j\rangle_{L^2(\Gamma)} \\ &\leqslant (2-\varepsilon)\left\|\sum_{i=1}^{n} q_0^{(i)}\phi_i\right\|^2_{L^2(\Gamma)}.\end{aligned} \tag{3.13}$$

On the other hand we have

$$\left\| \sum_{i=1}^{n} q_0^{(i)} \phi_i \right\|_{L^2(\Gamma)}^2 = \int_0^1 \left| \sum_{i,j=1}^{n} q_0^{(i)} \phi_i \right|^2 \mathrm{d}t$$

$$= \sum_{i,j=1}^{n} q_0^{(i)} q_0^{(j)} \int_0^1 \phi_i(t)\phi_j(t)\,\mathrm{d}t = (M_n q_0, q_0). \tag{3.14}$$

Insert (3.14) into (3.13) and we get

$$\varepsilon(M_n q_0, q_0) \leqslant (C_n q_0, q_0) \leqslant (2-\varepsilon)(M_n q_0, q_0). \tag{3.15}$$

Now we can formulate the following lemma:

Lemma 2. *Let λ_i denote the real and positive eigenvalues of the mass matrix M_n. Under the assumption*

$$\left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| \leqslant c \tag{3.16}$$

with a positive constant c it follows

$$\lambda_{\min}(q_0, q_0)_{M_n} \leqslant (M_n q_0, q_0) \leqslant \lambda_{\max}(q_0, q_0)_{M_n}, \tag{3.17}$$

which leads immediately to the estimate

$$\lambda_{\min}\|q_0\|_{L^2(\Gamma)}^2 \leqslant (M_n q_0, q_0) \leqslant \lambda_{\max}\|q_0\|_{L^2(\Gamma)}^2. \tag{3.18}$$

With the help of Eq. (3.15) we finally get

$$\varepsilon\lambda_{\min}\|q_0\|_{L^2(\Gamma)}^2 \leqslant (C_n q_0, q_0)_{M_n} \leqslant (2-\varepsilon)\lambda_{\max}\|q_0\|_{L^2(\Gamma)}^2. \tag{3.19}$$

The condition number $\kappa(C_n)$ of the matrix C_n can then be estimated to

$$\kappa(C_n) \leqslant \frac{(2-\varepsilon)\cdot\lambda_{\max}}{\varepsilon\cdot\lambda_{\min}}. \tag{3.20}$$

Using (3.16) we get

$$\kappa(C_n) \leqslant c\cdot\frac{2-\varepsilon}{\varepsilon}.$$

Theorem 1. *For the positive definite matrix C_n the conjugate gradient method converges and fulfils the following error estimate*:

$$\|e^{(i)}\|_{C_n} \leqslant 2\left(\frac{(\kappa(C_n)-1)^{\frac{1}{2}}}{(\kappa(C_n)+1)^{\frac{1}{2}}}\right)^{(i)} \|e^{(0)}\|_{C_n}, \tag{3.21}$$

with

$$\|e^{(i)}\|_{C_n} = \|q_{0,n}^{(i)} - q_{0,n}\|_{C_n} \quad \text{and} \quad \|e^{(0)}\|_{C_n} = \|q_{0,n}^{(0)} - q_{0,n}\|_{C_n}.$$

4. Numerical example

The mass matrix M_n and the right-hand side f_n in Eq. (3.5) can either be calculated analytically exact for special geometries or numerical integration is carried out. To keep the numerical integration error small, we handle the weak singularity of the integral kernel in the case of a non-smooth boundary by employing double partial integration, see [8,9,14,15].

Since we are considering a non-convex geometry, the main problem is the efficient detection of the shadow zones to calculate the visibility function $V(x,y)$ appearing as part of the stiffness matrix S. To reduce the computational effort, a geometrical algorithm has been developed in [10] to determine the shadow function in the two-dimensional case for polygonal domains. This algorithm has been transformed to the three-dimensional case for an enclosure with polyhedral boundary and consists of the following steps:

First we decide whether the geometry is convex or non-convex using an angle criterion. Then an element-orientated prestiffness matrix is calculated to reduce the number of elements we have to deal with in the last step, the nodewise calculation of the view factors. With this algorithm we obtain reasonable results since >5% of all view factors have to be calculated numerically. For more details see [15]. In the convex as well as in the non-convex case of a two-dimensional enclosure geometry the linear system (3.5) has been solved by some iterative methods, for example the cg-method with or without preconditioning, direct solvers or multigrid-methods, see [11]. However, in the three-dimensional case from our experiences the cg-algorithm with preconditioning has turned out to be the most efficient method and therefore has been implemented in our example.

As a non-convex geometry we consider an aperture as depicted and use a quadrangular discretization of the surface Γ into 480 elements. The emissivity coefficient is chosen as $\varepsilon = 0.2$, the Stefan–Boltzmann constant has the value $\sigma = 5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}^4}$ and the temperature source on the bottom is given by the function $T = 500\sqrt[4]{x(1.5-x)y(0.5-y)}\mathrm{K}$. The error is controlled a-posteriori by the residual.

Then the outgoing radiative flux q_0 looks as follows:

Acknowledgements

The first author of this paper would like to express his sincere thanks and appreciation to Prof. Dr. Gerald Warnecke for the support he received during

his stay at the Institute of Analysis and Numerics at Magdeburg University-Germany.

The second author would like to thank Prof. Dr. W.L. Wendland who gave her the possibility and the support to work in the area of heat radiation as member of the Sonderforschungsbereich SFB 259 "Hochtemperaturprobleme rückkehrfähiger Raumtransportsysteme" at the University of Stuttgart.

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