Mu'tah Journal For Research and Studies Vol. 11, No.6, 1996

On Lipschitz Functions of subnormal Operators

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Received on: 11/12/1996 Accepted on: 27/11/1996

الملخص
اقترانات ليبشتز للمؤثرات الطبيعية الجزئية
أثبتنا في هذا البحث النظرية التالية:
إذا كان S مؤثراً من النوع الطبيعي الجزئي يعمل على فضاء هلبرت H وكانت f اقتراناً تحقق شرط ليبشتز على جوار مفتوح يحتوي المجموعة الطيفية للمؤثر S فإن:
$\|f(S)X - Xf(S)\|_2 \leq k\|SX - XS\|_2$
حيث 0<k ، ولكل مؤثر X يعمل على H حيث $\|\|_2$ هو مقياس هلبرت-شميدت.
كذلك فإن البحث يحتوي على بعض النتائج والنظريات ذات العلاقة بالنظرية الرئيسية.

ABSTRACT

We prove the following:

Theorem: Let S be a subnormal operator on an infinite dimensional separable complex Hilbert space H. Let f be analytic and Lipschitz on a neighborhood E of the spectrum σ(S) of S. Then

$$\|f(S)X - Xf(S)\|_2 \leq k\|SX - XS\|_2,$$

for some $k > 0$ and for every $X \in B(H)$, the algebra of all bounded linear operators on H, where $\|\|_2$ is the Hilbert-Schmidt norm.

A.M.S. Subject classification number 1990: 47 B 20, 47 A 25

Key Words: Operator, Subnormal, Hilbert space, Hilbert-Schmidt norm, Lipschitz Function, Spectrum and Analytic Function.

INTRODUCTION

Let H denote an infinite dimensional complex Hilbert space, B(H) denote the algebra of all bounded linear operators on H. For a compact subset E of the complex plane C, let R(E) denote the algebra of all bounded analytic functions defined on E; and L(E) the set of all functions which satisfies Lipschitz condition on E, i,e., $f \in L(E)$ if and only if there is a real number $k > 0$ such that:

$$|f(z)-f(t)| \leq k|z-t|,$$

for all $z,t \in E$.

In [1], it is proved that if A is a self adjoint operator and $f \in L(\sigma(A))$ then

$$\|f(A)X - Xf(A)\|_2 \leq k\|AX - XA\|_2$$

for some k>0, and every $X \in C_2$, the class of Hilbert-Schmidt operators, where $\sigma(A)$ is the spectrum of A and $\|\cdot\|_2$ is the Hilbert-Schmidt norm, defined by

$$\|T\|_2 = \left(\sum_{n=1}^{\infty}\|Te_n\|^2\right)^{\frac{1}{2}},$$

where (e_n) is an orthonormal basis of H, and T is any operator on H.

In [6], this result has been generalized to normal operators as follows;

Theorem: If N is a normal operator on H, and $f \in L(\sigma(N))$ then,

$\|f(N)X - Xf(N)\|_2 \leq k\|NX - XN\|_2$,

for some $k > 0$, and every $X \in B(H)$, where an operator N is said to be normal if $NN^* = N^*N$, (The operator N^* is the adjoint of N).

For the proof see [6].

In this article we generalize the above theorem to subnormal operators on H. An operator S on H is called subnormal if there is a normal extension N on a Hilbert space $K \supset H$. If N is a minimal normal extension of S, i.e., if N has no reducing subspace L of K which contains H properly; then N has the following representation

$$N = \begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix}, \quad (1)$$

on $K = H \oplus H^{\perp}$, where A is some operator from $H^{\perp}$ to H. In this case T is called the dual of S, (and T^* is the adjoint of T).

A subset E of the complex plane is called a spectral set of an operator $S \in B(H)$ if and only if, the spectrum $\sigma(S)$ of S is a subset of E and

$$\|f(S)\| \le \|f\|_\infty = \sup\{|f(z)| : z \in E\},$$

for every rational function f with poles off E, or equivalently $f \in R(E)$.

Conway [4] proved that if S is a subnormal operator with dual T and E is a spectral set of S then,

$$f(N) = \begin{bmatrix} f(S) & A' \\ 0 & f(T^*) \end{bmatrix}, \tag{2}$$

for any $f \in R(E)$, where A' is some operator from $H^\perp$ to H.

The following theorem is the main result of this article.

Theorem 1: Le S be a subnormal operator on H. If $f \in R(E) \cap L(E)$, then

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2 \tag{3}$$

for some $k>0$, and every $X \in B(H)$.

Proof: Since S is a subnormal operator on H, therefore there is a sequence of normal operators (N_i) such that (N_i) converges to S, in the strong operator topology, denoted by sot; (see Conway [3] pages 123, 124, 125).

We claim that if $f \in R(E)$, then $f(N_i)$ converges to $f(S)$ (sot). Indeed; by the construction of (N_i),

$$N_i = v_i^{-1} N v_i, \ I=1,2,3,\ldots, \tag{4}$$

where N is a normal extension of S to a Hilbert space $K \supset H$, (See the proof of Theorem 1.17, page 125 in Conway [3]), and

v_i: $H \to K$,

is an isomorphism such that v_i=identity on H_i, where

$H_i = V\{e_1,\ldots,e_i\ Se_1,\ldots Se_i\} = \text{Span}\{e_i,\ldots,e_i,Se_1,\ldots,Se_i\}$.

This means that N_i and N are similar (where two operators S,T are said to be similar if there is an invertible operator $X{:}S=X^{-1}TX$). Thus, by problem 75 in Halmos [5], (page 42, which says that two similar operators has the same spectrum).

$$\sigma(N_i) = \sigma(N), \ i=1,2,3,\ldots$$

Since N is a normal extension of S, it follows that $\sigma(N) \subset \sigma(S)$. Therefore

$\sigma(N_i) \subset E$, i=1,2,3,...

Thus $f(N_i)$ is well-defined for every f in R(E) and every i=1,2,3,...

Now, since $f \in R(E)$, there is a sequence of polynomials (P_n) such that (P_n) converges to f, uniformly on compact subsets of E. This implies, using Conway [2] (page 206, by Riesz functional calculus theorem) that $P_n(N_i)$ converges to $f(N_i)$, in the uniform operator topology (denoted by (uot).), i.e., given i, we have:

$$\|P_n(N_i) - f(N_i)\| \to 0,\ n \to \infty, i = 1,2,3,\ldots$$
$$\|P_n(S_i) - f(S)\| \to 0,\ n \to \infty.$$

Thus, for every $\in > 0$, and nonzero $e \in H$, there is J such that:

$$\|P_n(N_i) - f(N_i)\| < \frac{\in}{3\|e\|},\ n > J,\ i = 1,2,3,\ldots$$

$$\|P_n(S_i) - f(S)\| < \frac{\in}{3\|e\|},\ n > J.$$

For arbitrary $e \in H$, and n_0=J+1, this implies that:

$$\begin{aligned} \|(f(N_i) - f(S))e\| &\le \|(f(N_i) - P_{n_0}(N_i))e\| + \|(P_{n_0}(N_i) - P_{n_0}(S))e\| \\ &+ \|(P_{n_0}(S) - f(S))e\| \qquad (5) \\ &< 2\in/3 + \|(P_{n_0}(N_i) - P_{n_0}(S))e\|. \end{aligned}$$

Since P_{n_o} is a polynomial and since the strong convergence is sequentially continuous (see Halmos [5], problem, 113 page 62), one concludes that

$$P_{n_o}(N_i) \rightarrow P_{n_o}(S), \text{ as } i \rightarrow \infty, \text{ (sot)},$$

and thus to every $\in>0$, $\exists$ J such that:

$$\left\|\left(P_{n_0}(N_i) - P_{n_0}(S)\right)e\right\| < \in/3, \text{ for } i > J. \qquad (6)$$

Combining (5) and (6) gives the convergence of $f(N_i)$ to $f(S)$, (sot), for every $f \in R(E)$, (notice that this result is trivial for e=0).

Now, applying problem 112 page 62 in Halmos [5], (which inserts that product is left and right continuous with one argument fixed, $A \rightarrow AB$ or $A \rightarrow BA$) one obtains the following:

$$N_iX \rightarrow SX, \text{ sot}; XN_i \rightarrow XS, \text{ sot}, i \rightarrow \infty,$$

$$f(N_i)X \rightarrow f(S)X, \text{ sot}; Xf(N_i) \rightarrow Xf(S), \text{ sot}, i \rightarrow \infty \quad ,$$

for an arbitrary X in B(H). This implies that:

$$N_iX\text{-}XN_i \rightarrow SX\text{-}XS, \text{ sot}, i \rightarrow \infty, \qquad (7)$$

$$f(N_i)X\text{-}Xf(N_i) \rightarrow f(S)X\text{-}Xf(S), \text{ sot}, i \rightarrow \infty \quad , \qquad (8)$$

In [6], we have, for every $\in>0$, $N=D_\in+K_\in$ and $f(N)=f(D_\in)+C_\in$, where $D_\in$ is a diagonal operator, $\|K_\in\|_2<\in, \|C_\in\|_2 \to 0$, as $\in\to 0$.

If $D_\in e_n=\lambda_n e_n$, and $X=(x_{ij})$, is the corresponding matrix of X, relative to the orthonormal basis (e_n) of H, then the (ij) entry for $D_\in X-XD_\in$ is $(\lambda_i-\lambda_j)x_{ij}$. Also the (ij) entry for $f(D_\in)X-Xf(D_\in)$ is $\left(f(\lambda_i)-f(\lambda_j)\right)x_{ij}$.

Hence,

$$\begin{aligned}\left|\|(NX-XN)e_n\|-\|(D_\in X-XD_\in)e_n\|\right| &\le \|(K_\in X-XK_\in)e_n\|\\ &\le 2\|K_\in\|_2\ \|X\|,\end{aligned}$$

and

$$\begin{aligned}\left|\|(f(N)X-Xf(N))e_n\|-\|(f(D_\in)X-Xf(D_\in))e_n\|\right| &\le \|(C_\in X-XC_\in)e_n\|\\ &\le 2\|C_\in\|_2\ \|X\|.\end{aligned}$$

Moreover,

$$\begin{aligned}\|(f(D_\in)X-Xf(D_\in))e_n\|^2 &= \sum_{i=1}^{\infty}|f(\lambda_i)-f(\lambda_n)|^2|x_{in}|^2\\ &\le k^2\sum_{i=1}^{\infty}|\lambda_i-\lambda_n|^2|x_{in}|^2 \le k^2\|(D_\in X-XD_\in)e_n\|^2.\end{aligned}$$

Letting $\in\to 0$, we obtain:

$$\|(f(N)X - Xf(N))e_n\| \le k\|(NX - XN)e_n\|, \tag{9}$$

for every n.

Having obtained these results we have,

$$\big|\|(SX - XS)e_n\| - \|(N_iX - XN_i)e_n\|\big| \le \|[(SX - XS) - (N_iX - XN_i)]e_n\| \to 0$$

as $i \to \infty$.

Since $f(N_i)$ converges to $f(S)$ in (sot) one has:

$$\begin{aligned}&\big|\|(f(S)X - Xf(S))e_n\| - \|(f(N_i)X - Xf(N_i))e_n\|\big| \\ &\qquad\qquad \le \|[(f(S)X - Xf(S)) - (f(N_i)X - Xf(N_i))]e_n\| \to 0\end{aligned}$$

as $i \to \infty$.

Combining these inequalities, for arbitrary $\in > 0$, e in H and sufficiently large i and using (9), one obtains:

$$\begin{aligned}\|(f(S)X - Xf(S))e_n\| &\le \|(f(N_i)X - Xf(N_i))e_n\| + \in/2 \\ &\le k\|(N_iX - XN_i)e_n\| + \in/2 \\ &< k\left\{\|(SX - XS)e_n\| + \frac{\in}{2k}\right\} + \in/2 \\ &< k\|(SX - XS)e_n\| + \in .\end{aligned}$$

Thus (since $\in$ is arbitrary),

$$\|(f(S)X - Xf(S))e_n\| \le \|(SX - XS)e_n\| \,, \tag{10}$$

Therefore,

$$\begin{aligned} \|f(S)X - Xf(S)\|_2^2 &= \sum_{n=1}^{\infty} \|(f(S)X - Xf(S))e_n\|^2 \\ &\le \sum_{n=1}^{\infty} k^2 \|(SX - XS)e_n\|^2 \\ &\le k^2 \|SX - XS\|_2^2 \end{aligned}$$

which ends the proof.

Remark: The analyticity of the Lipschitz function in Theorem 1 is essential, this is emphasized by the following.

Example: Let S be the unilateral shift on H, defined by $Se_n=e_{n+1}$, n=1,2,... and U be the bilateral shift on H, defined by $Ue_n=e_{n+1}$, n=..., -1, 0, 1, 2,... and let $f(z) = \bar{z}$. This function is Lipschitz with k=1, but not analytic on a neighborhood of $\sigma(S)$, which is the closed unit disc $\bar{D}$. It is known that S is subnormal (but not normal), with adjoint $S^* = f(S)=f(U)|_H=U^*|_H$, where U is the bilateral shift; a normal extension of S to $H \oplus H^{\perp}$, see [3]. If we assume that Theorem 1 is true for non analytic Lipschitz function, and we let X=S, we would obtain that:

$$\|S^*S - SS^*\| \leq \|SS - SS\| = 0,$$

which means that S is normal, which is a contradiction.

Corollary: Let S_1, S_2 be subnormal operators in B(H) and let $f \in R(E) \cap L(E)$, $E \supset \sigma(S_1) \cup \sigma(S_2)$ then:

$$\|f(S_1)X - Xf(S_2)\|_2 \leq k\|S_1X - XS_2\|_2,$$

for some k>0, and every X in B(H). If X=I, then

$$\|f(S_1) - f(S_2)\|_2 \leq k\|S_1 - S_2\|_2$$

Proof: Define S and Y on $H \oplus H$ by:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

The operator S is subnormal and for any $f \in R(E)$,

$$f(s) = \begin{bmatrix} f(S_1) & 0 \\ 0 & f(S_2) \end{bmatrix},$$

Therefore, we have:

$$\|SY - YS\|_2 = \|S_1X - XS_2\|_2,$$

and

$$\|f(S)Y - Yf(S)\|_2 = \|f(S_1)X - Xf(S_2)\|_2.$$

Applying Theorem 1 to S, for $f \in R(E) \cap L(E)$, we obtain the conclusion, and the special case follows by taking X=I.

Theorem2: If S is a subnormal operator on H, whose dual is T on $H^\perp$ and *if* f is in $R(E) \cap L(E)$, where $\sigma(S) \subset E$, then,

$$\|f(S)X - Xf(T^*)\|_2 \le k\|SX - XT^*\|_2,$$

for some k>0, and every $X \in B(H^\perp, H)$; the algebra of bounded linear operators from $H^\perp$ to H. In particular, if $H^\perp$ is a copy of H, (i.e., $H^\perp \cong H$), then:

$$\|f(S) - f(T^*)\|_2 \le k\|S - T^*\|_2,$$

Proof: for $X \in B(H^{\perp},H)$, define $Y \in B(K)$, $K=H \oplus H^{\perp}$ by,

$$Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Using the representations (1) one can show that:

$$NY - YN = \begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix}\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix} = \begin{bmatrix} 0 & SX - XT^* \\ 0 & 0 \end{bmatrix}.$$

Similarly for $f \in R(E) \cap L(E)$, and using (2), one obtains:

$$f(N)Y - Yf(N) = \begin{bmatrix} 0 & f(S)X - Xf(T^*) \\ 0 & 0 \end{bmatrix}.$$

For an orthonormal basis (e_n) for $H^{\perp}$, we obtain:

$$(NY - YN)\begin{bmatrix} 0 \\ e_n \end{bmatrix} = \begin{bmatrix} (SX - XT^*)e_n \\ 0 \end{bmatrix}, \tag{11}$$

and

$$\left(f(N)Y - Yf(N)\right)\begin{bmatrix} 0 \\ e_n \end{bmatrix} = \begin{bmatrix} \left(f(S)X - Xf(T^*)\right)e_n \\ 0 \end{bmatrix}. \tag{12}$$

Combining (11), (12) and applying (9), one obtains:

$$\|(f(S)X - Xf(T^*))e_n\| = \|(f(N)Y - Yf(N))e_n\|$$
$$\leq k\|(NY - YN)e_n\|$$
$$\leq k\|(SX - XT^*)e_n\|,$$

for every n, and thus:

$$\|f(S)X - Xf(T^*)\|_2^2 = \sum_{n=1}^{\infty}\|(f(S)X - Xf(T^*))e_n\|^2$$
$$\leq k^2\sum_{n=1}^{\infty}\|(SX - XT^*)e_n\|^2$$
$$\leq k^2\|(SX - XT^*)e_n\|_2^2 .$$

This implies that:

$$\|f(S)X - Xf(T^*)\|_2 \leq k\ \|SX - XT^*\|_2,$$

for the other conclusion let X=I on $H^{\perp}$=H, and this proves the theorem.

Two operators T, S on H are said to be unitarily equivalent if there is a unitary operator u on H such that:

$$T=uSu^* .$$

For unitarily equivalent operators we prove the following:

Theorem3: Let $S\in B(H)$, $f\in R(E)\cap L(E)$, where $\sigma(S)\subset E$, such that

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2 \tag{13}$$

for some $k>0$, and any $X\in B(H)$. If $T\in B(H)$ is unitarily equivalent to S, then

$$\|f(T)Y - Yf(T)\|_2 \le k\|TY - YT\|_2,$$

for some $k>0$, and any $Y\in B(H)$.

Proof: Let u be a unitary operator on H, such that $T=uSu^*$. Since $f\in R(E)$, it is not difficult to show that $f(T)=uf(S)u^*$. If (e_n) is an orthonormal basis for H, it is known that (u^*e_n) is an orthonormal basis for H too, for any unitary operator u. Thus one has

$$\begin{aligned}\|(f(T)Y - Yf(T))e_n\| &= \|(uf(S)u^*Y - Yuf(S)u^*)e_n\| \\ &= \|(uf(S)u^*Yuu^* - uu^*Yuf(S)u^*)e_n\| \\ &= \|(uf(S)X - Xf(S))u^*e_n\| \\ &= \|(f(S)X - Xf(S))g_n\|,\end{aligned} \tag{14}$$

where

$X=u^*Yu$, $Y\in B(H)$, $(g_n)=(u^*e_n)$.

Using (14), we have:

$$\begin{aligned}\|f(T)Y - Yf(T)\|_2^2 &= \sum_{n=1}^{\infty}\|(f(T)Y - Yf(T))e_n\|^2\\ &= \sum_{n=1}^{\infty}\|(f(S)X - Xf(S))g_n\|^2\\ &= \|f(S)X - Xf(S)\|_2^2 \ .\end{aligned}$$

By the assumption of the theorem one concludes that

$$\begin{aligned}\|f(T)Y - Yf(T)\|_2^2 &\le k^2\|SX - XS\|_2^2\\ &\le k^2\sum_{n=1}^{\infty}\|(SX - XS)g_n\|^2\\ &\le k^2\sum_{n=1}^{\infty}\|u(SX - XS)u^*e_n\|^2\\ &\le k^2\sum_{n=1}^{\infty}\|(uSu^*uXu^* - uXu^*uSu^*)e_n\|^2\\ &\le k^2\sum_{n=1}^{\infty}\|(TY - YT)e_n\|^2\\ &= k^2\ \|(TY - YT)\|_2^2\ ,\end{aligned}$$

which shows that

$$\|f(T)Y - Yf(T)\|_2 \le\ k\ \|TY - YT\|_2\ .$$

To end up the proof we should remark that since T is unitarily equivalent to S, they have the same spectrum, so that $\sigma(T)\subset E$ as well.

Finally we end this article by the following.

Theorem:4: If S satisfies (13), then every operator in the closure of the unitary orbit u(S) of S in the uniform operator topology (uot) satisfies (13), where

$$u(S)=\{uSu^{*}: u \text{ is unitary on } H\}.$$

Proof: Since $S\in B(H)$ such that for $f\in R(E)\cap L(E)$, we have

$$\|f(S)X - Xf(S)\|_2 \leq k\,\|SX - XS\|_2, \tag{15}$$

for some k>0, and every X in B(H). Theorem 3 shows that every operator in u(S) satisfies (13). So that, we assume that (u_n) is a sequence of unitary operators on H, such that $(u_nSu^*_n)$ converges to A in (uot), and we want to show that A satisfies (13). Indeed, from uniform convergence: $\|u_nSu_n^* - A\| \to 0$, we have $(u_nf(S)u^*_n)$ converges to f(A) in (uot), for every $f\in R(E)$. By the upper semi continuity of the spectrum, one has

$$\sigma(f(A)) \subset \sigma(u_n f(S) u^*_n) \subset E.$$

Since uniform operator topology convergence of (T_i) to A implies uniform operator topology convergence of $f(T_i)$ to $f(A)$, for $f \in R(E)$, one concludes for unitary (u_n) that

$$(u_n f(S) u^*_n) = f(u_n S u^*_n),$$

converges to f(A) in (uot). Now we have

$$\begin{aligned}
\|(f(A)X - Xf(A))e\| &\leq \|(f(A)X - Xf(A)) - ((u_n f(S)u_n^*)X - X(u_n f(S)u_n^*))e\| \\
&\quad + \|((u_n f(S)u_n^*)X - X(u_n f(S)u_n^*))e\| \\
&\leq \in/2 + \|u_n(f(S)Y_n - Y_n f(S)u_n^*)e\| \\
&\leq \in/2 + \|(f(S)Y_n - Y_n f(S))g\| \\
&\leq \in/2 + k\|(SY_n - Y_n S)g\| \\
&\leq \in/2 + k\|(u_n S u_n^* X - X u_n S u_n^*)e\| \\
&\leq \in/2 + k\|(u_n S u_n^* X - X u_n S u_n^*)e - (AX - XA)e\| \\
&\qquad + k\|(AX - XA)e\| \\
&\leq \in/2 + .k\left(\frac{\in}{2k}\right) + k\|(AX - XA)e\| \\
&\leq \in + k\|(AX - XA)e\| .
\end{aligned}$$

Since $\in$ is arbitrary, we have

$$\|(f(A)X - Xf(A))e\| \le k\|(AX - XA)e\|.$$

Now, for an orthonormal basis (e_n) for H,

$$\begin{aligned}\|f(A)X - Xf(A)\|_2^2 &= \sum_{n=1}^{\infty}\|(f(A)X - Xf(A))e_n\|^2 \\ &\le k^2\sum_{n=1}^{\infty}\|(AX - XA)e_n\|^2 \\ &\le k^2\|AX - XA\|_2^2,\end{aligned}$$

and thus A satisfies (13) for every A in the uniform closure of u(S).

Acknowledgements

I would like to thank to referees for their useful suggestions.

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