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On Existence and Uniqueness Theorem Concerning Time–Dependent Heat Transfer Model

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Abstract

In this article we consider a physical model describing time–dependent heat transfer by conduction and radiation. This model contains two conducting and opaque materials which are in contact by radiation through a transparent medium bounded by diffuse–grey surfaces. The aim of this work is to present a reliable framework to prove the existence and the uniqueness of a weak solution for this problem. The existence of the solution can be proved by solving an auxiliary problem by the Galerkin–based approximation method and Moser–type arguments which implies the existence of solution to the original problem. The uniqueness of the solution will be proved by using the same approach in our previous work for the stationary heat transfer model and some ideas from nonlinear heat conduction analysis.

Keywords: Heat transfer modes, Galerkin approximation method, existence and uniqueness of physical solutions

AMS 2000 Mathematics Subject Classification Numbers: 35J65, 45P05

1. Introduction

Heat radiation plays a significant role in heat transfer in various cases, typically when a hot surface is in contact with a transparent or semitransparent medium with relatively low heat conductivity. In fact, these conditions are often met already at room temperatures–why else would we call radiators by the name. In this paper we are going to study a heat radiation model that is, in some sense, about the simplest non–trivial case of conductive body with nonconvex opaque radiating surface.

Before we can present the model, some background in the physics of heat radiation must be illustrated. In typical situation involving heat radiation, we have to combine radiation with other heat transfer mechanisms (conduction, convection) that are very slow compared to the speed of light. Hence, to a very good approximation, the radiactive heat transfer is instantaneous and can be modeled using stationary equations. In principle we have to solve the transfer equations at any point for each direction and wavelength of the rays. However, to simplify the situation we make the following important assumptions: We restrict ourselves to grey materials, which are bounded with grey and diffuse walls. This means that we can forget the angular and spectral distribution of radiation and model only the total intensities.

In our previous work on heat radiation (see for example Bialecki (1993), Qatanani (2003), Qatanani and Barghouthi (2005), Qatanani and Schulz (2006)) the radiosity equation has received very much attention. There, we have focused on both the theoretical and the numerical aspects of this equation. Moreover, the problem of coupling radiation with other heat modes was also studied by many authors (see for example Bialecki (1993), Laitinen and Tiihonen (2001), Modest (1993), Qatanani (2007), Qatanani and Salah (2005), Qatanani et al. (2007)).

Concerning the simplest nontrivial case of conductive body with nonconvex opaque radiating surface, we are a ware of the work (see Kelley (1996), Zeidler (1986)) and our previous work (see Qatanani (2007), Qatanani and Salah (2005)). They all studied some properties of the operators related to the radioactive transfer and showed the existence of a weak solution under some restrictions (no enclosed surfaces, limitations to material properties). In the case of semitransparent material the analysis has been carried out in one dimensional case with non reflecting surfaces (see Kelley (1996)) and in two and three–dimensional with diffusively reflecting surfaces (see Qatanani (2007)).

The present work is an extension to our previous work (see Qatanani et al. (2007)) on the existence and the uniqueness of the solution of the heat transfer stationary model which is an abstraction of contactless heat transfer in protected environment arising for example in semiconductor applications. To this end, we will investigate the existence and the uniqueness of the time–dependent counterpart of the stationary model. This physical model describing heat transfer by conduction and radiation will be presented in section 2.

The existence of a solution for this model will be proved by solving an auxiliary problem by the Galerkin method. The uniqueness of the solution will be proved by similar approach used for the stationary case and some ideas borrowed from the analysis of nonlinear heat conduction. Throughout this work we will use the following notations:

(i) The duality between L^p_μ and L^q_μ for a Borel measure μ is defined as

$$\langle f, g \rangle = \int f \, g \, d\mu, \qquad f \in L^p_\mu \text{ and } g \in L^q_\mu$$

with $1 \le p \le \infty$, p and q are conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$.

(ii) An operator K is positive if $f \ge 0$ implies $K\,f \ge 0$.

We denote the positive and negative parts of a function by either sub–or superscript:

$$f^{+} = f_{+} = \max(f,0) \quad \text{and} \quad f^{-} = f_{-} = \max(-f,0).$$

(iii) Let Γ be a subset of $\partial\Omega$ where local heat transfer occurs and define an operator A through

$$\langle A f, g \rangle = \int_{\Omega} a_{ij}\,\partial_i f\,\partial_i g\,dx + \int_{\Gamma} \xi |f|^{p-1} f\,g\,ds\,, \quad p > 1,$$

where the coefficients a_{ij} *and* $\xi \ge 0$ are bounded. The domain of A is

$$H^1(\Omega) \cap L_{\gamma}^{p+1}(\Gamma),$$

where the measure γ is the surface measure of Γ weighed with the coefficient ξ. The null space of A is denoted by

$$N(A) = \left\{ f \in H^1(\Omega) \cap L_{\gamma}^{p+1}(\Gamma) : A f = 0 \right\}.$$

(iv) $\{ a_{ij} \}$ is strictly elliptic, that is, there exists a constant $C > 0$ such that

$$\langle A f, f \rangle \ge C \int_{\Omega} |\nabla f|^2\,dx \quad \text{for all} \quad f \in H^1(\Omega).$$

(v) We denote by $C > 0$ different positive constants appearing in the proofs.

2. The Physical Model

Let $\Omega = \Omega_1 \cup \Omega_2 \subset R^3$ be a union of two disjoint, conductive and opaque enclosures surrounded by transparent and non–conductive medium. Moreover, suppose that the radioactive surfaces Γ_1 and Γ_2 are diffuse and grey, that is, the emissivity ϵ of these surfaces depends neither on direction nor on the wavelength of radiation. Under the above assumptions the boundary value problem reads as

$$-\nabla.(k\nabla T) = g \quad \text{in} \quad \Omega_1 \cup \Omega_2, \tag{2.1}$$

$$-k\frac{\partial T}{\partial n} = q \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{2.2}$$

where k is the heat conductivity, n is the outward unit normal, g is the given heat generation distribution and q is the radiative heat flux, which is defined as the difference between the outgoing radiation q_o and the incoming radiation q_i. ϵ is the emissivity coefficient ($0 < \epsilon < 1$) and T is absolute temperature. For convex Γ_2 we can assume the external radiation is given. Therefore, the radiative flux on Γ_2 consist only from local emission given by Stefan–Boltzmann Law,

$$q\big|_{\Gamma_2} = \epsilon\,\sigma(T^4 - T_0^4),$$

where σ is the Stefan–Boltzmann constant which has the value $5.669996\times 10^{-8}\, W/(m^2 K^4)$ and T_0^4 is the effective external radiation temperature. The surface Γ_1, however, receives radiation from other parts of itself, leading to the relation

$$q_i(x) = \left(Q\, q_o\right)(x) \quad \text{on } \Gamma_1, \tag{2.3}$$

where Q is an integral operator with kernel defined on $\Gamma_1 \times \Gamma_1$. Moreover, we note that the outgoing radiation q_o on Γ_1 is a combination of emission and reflected fraction of incoming radiation (see Modest (1993)), that is,

$$q_o = \epsilon\sigma T^4(x) + (1\text{-}\epsilon)\; q_i(x) = \epsilon\sigma T^4(x) + (1\text{-}\epsilon) Q\, q_i(x). \tag{2.4}$$

Solving, for q_o as a function of T we can write our problem in a variational form as

$$\int_{\Omega} k\,\nabla T\,\nabla\psi\,dx + \int_{\Gamma_1} (G\,\sigma\,T^4)\psi\,ds + \int_{\Gamma_2} \epsilon\sigma\,T^4\psi\,ds = \int_{\Omega} g\,\psi\,dx + \int_{\Gamma_2} \epsilon\sigma\,T^4\psi\,ds, \tag{2.5}$$

where the operator G is defined as

$$GT = \left(I - Q\right)\left(I - (1-\epsilon)\,Q\right)^{-1} \epsilon\, T.$$

In stationary case our problem reads as: given $g \in X^*$, find $T \in X$ such that

$$\langle KT,\psi\rangle = \langle AT,\psi\rangle + \int_{\Gamma_1} (G\,|T|^3 T)\psi\, d\mu = \langle g,\psi\rangle, \quad \forall \psi \in X, \tag{2.6}$$

where the solution space is given by $X = H^1(\Omega)\cap L_\mu^5(\Gamma_1)\cap L_\gamma^{p+1}(\Gamma_2)$. One observes that since the Stefan–Boltzmann law is physical only for nonnegative values of temperature, we can replace the term T^4 by $|T|^3 T$ for mathematical convenience. In fact problem (2.6) has already been investigated in our recent work (see Qatanani et al. (2007)). In the current work our main goal is to investigate the time dependent counterpart problem of (2.6). That is, for $g \in V^*$ and $T_0 \in L^2(\Omega)$, we seek $T \in W$ such that

$$\langle T'(t),\psi\rangle_X + \langle KT(t),\ \psi\rangle_X = \langle g(t),\ \psi\rangle_X, \tag{2.7}$$

$$T(0) = T_0, \tag{2.8}$$

for all $\psi \in X$ and almost all $t \in [0,\tau]$. The operator *K* is defined as in (2.6). For simplicity we assume $p = 1$ on Γ_2. This problem is nontrivial since for the Galerkin method we need an a priori estimate in the space $L^5\left(0,\tau,L_\mu^5\right)$ which does not follow from the coercivity of the operator $K : X \to X^*$ (see Qatanani et al. (2007)).

Metzger (1999) solved this problem for radiative systems without enclosure. However, our approach here is to assume more regularity from the data which allows us to derive the a priori estimate using Moser iteration (see Clement, Zacher (2008)). Our idea is to solve first an auxiliary problem for which the a priori estimate in $L^5\left(0,\tau,L_\mu^5\right)$ trivially holds: Fix $\varepsilon > 0$ and seek T_ε such that

$$\langle T_\varepsilon'(t),\psi\rangle_X + \langle KT_\varepsilon(t),\psi\rangle_X + \varepsilon\int |T_\varepsilon(t)|^3 T_\varepsilon(t)\psi\, d\mu = \langle g(t),\psi\rangle_X, \tag{2.9}$$

$$T(0) = T_0, \tag{2.10}$$

for all $\psi \in X$ and almost all $t \in (0,\tau)$. Then, we will prove with Moser-type arguments that the auxiliary problem is in $L^5(0,\tau,L_\mu^5)$ independent of $\varepsilon > 0$ which allows us to deduce that T_ε converges to the solution of the original problem.

Let us introduce first some notations and outline the basic properties of the function spaces V, V^* and *W*. The spaces *V* and *W* are defined as

$$V = L^2\left(0,\tau,H^1(\Omega)\right)\cap L^5\left(0,\tau,L^5_\mu\right)$$

$$W = \left\{T : T\in V,\ T'\in V^*\right\}.$$

Throughout this work we will use the abbreviations:

$$\langle T,\psi\rangle = \langle\psi,T\rangle_X\ ,\quad (T,\psi)=(\psi,T)_{L^2(\Omega)}\ ,\ L^p(B)=L^p(0,\tau,B)\,.$$

The function space *V* is equipped with the norm

$$\|T\|_V = \|T\|_{L^2(H^1(\Omega))} + \|T\|_{L^5(L^5_\mu)}\,.$$

To define the norm for V^*, we observe that each $v\in V^*$ can be expressed as $v=T+\psi$ with $T\in L^2\left(H^1(\Omega)^*\right)$ *and* $\psi\in L^{5/4}\left(L^{5/4}_\mu\right)$; (see for example Krizek and Liu (1996)). Hence, we can define

$$\|v\|_{V^*} = \inf_{v=T+\psi}\left\{\|T\|_{L^2(H^1(\Omega)^*)} + \|\psi\|_{L^{5/4}(L^{5/4}_\mu)}\right\}$$

and finally, we define the norm of *W* as

$$\|T\|_W = \|T\|_V + \|T'\|_{V^*}\,.$$

The most important properties of the space *W* are summarized in the following lemma:

Lemma 1: The spaces $V,\ V^*$ and W are reflexive Banach spaces. Moreover, the embedding $W\subset C\left(\left[0,\tau\right],L^2(\Omega)\right)$ is continuous and the following result holds

$$\left(T(t),\psi(t)\right)-\left(T(x),\psi(x)\right)=\int_x^t \langle T'(y),\psi(y)\rangle+\langle\psi'(y),T(y)\rangle\,dy\,,$$

for all $T,\psi\in W$ and $x,t\in\left[0,\tau\right],\ x\le t$.

The most important properties of the stationary operator $K : X\to X^*$ also extend to the time–dependent case.

Lemma 2: The operator K: $V \to V^*$ is bounded and pseudomonotone.

Proof: From the boundedness of *A* and *G* (see Qatanani et al. (2007)) together with the Hölder inequality it follows that there is a constant $c > 0$ such that

$$\| K\, T \|_{V^*} \le c \left\{ \| T \|_{L^2(H^1(\Omega))} + \| T \|^4_{L^5(L^5_\mu)} \right\} \quad \forall\, T \in V\ .$$

Moreover, $A{:}V \to V^*$ is monotone because $A{:}X \to X^*$ is monotone.

To prove the pseudomonotonicity of *K*, we assume that T_i converges weakly to *T* in *V*. Then, $\left|T_j\right|^3 T_j$ converges weakly to $\left|T\right|^3 T$ in $L^{5/4}(0,\tau,L^{5/4}_\mu)$ by the arguments in (see Qatanani et al. (2007)). Hence,

$$\lim_{j\to\infty} \int_0^\tau \langle G\left|T_j\right|^3 T_j\,,\psi\rangle_\mu\, dt = \int_0^\tau \langle G\left|T\right|^3 T,\psi\rangle_\mu\, dt,\ \forall \psi \in V.$$

Hence, the Fatou's lemma yields

$$\begin{aligned}\underline{\lim}_{j\to\infty} \langle G\left|T_j\right|^3 T_j\,,T_j - \psi\rangle_V \ge\ & \underline{\lim}_{j\to\infty} \int_0^\tau \langle G\left|T_j\right|^3 T_j,T_j\rangle_\mu\, dt - \overline{\lim}_{j\to\infty} \int_0^\tau \langle G\left|T_j\right|^3 T_j,\psi\rangle_\mu\, dt \\ \ge\ & \int_0^\tau \underline{\lim}_{j\to\infty} \langle G\left|T_j\right|^3 T_j,T_j\rangle_\mu\, dt\ - \lim_{j\to\infty} \int_0^\tau \langle G\left|T_j\right|^3 T_j,\psi\rangle_\mu\, dt \\ \ge\ & \int_0^\tau \langle G\left|T\right|^3 T,T\rangle_\mu\, dt\ - \int_0^\tau \langle G\left|T\right|^3 T,\psi\rangle_\mu\, dt,\end{aligned}$$

where the pseudomonotonicity $G\left|T\right|^3 T : X \to X^*$ has been used.

3. The Auxiliary Problem and Galerkin Approximation

In order to introduce the Galerkin equation for the auxiliary problem (2.9)–(2.10), let $\left\{v_1, v_2, \ldots\ldots., v_n\right\}$ be a linearly independent set in *X* and define $X_n = span\left\{v_1, v_2, \ldots\ldots., v_n\right\}$. Further, set

$$T_n(t) = \sum_{k=1}^{n} a_{kn}(t)\, v_k \ . \tag{3.1}$$

Then, we seek $T_n \in W$ such that

$$\langle T_n'(t), v_j \rangle + \langle K T_n(t), v_j \rangle + \varepsilon \int \left| T_n(t) \right|^3 T_n(t)\, v_j \, d\mu = \langle g(t), v_j \rangle \tag{3.2}$$

and

$$T_n(0) = T_{n_0} \in X_n \tag{3.3}$$

for $j = 1, \ldots\ldots\ldots, n$ and T_{n_0} is chosen such that

$$T_{n_0} \to T_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to \infty .$$

Theorem 1: Let $T_0 \in L^2(\Omega)$, $g \in V^*$ and assume that T_n is a solution of (3.2)–(3.3). Then there exists a constant $C > 0$ independent of n such that

$$\| T_n \|_V \le C \tag{3.4}$$

$$\| K T_n \|_{V^*} \le C \tag{3.5}$$

$$\max_{0 \le t \le \tau} \| T_n(t) \|_{L^2(\Omega)} \le C . \tag{3.6}$$

Proof: We start by multiplying (3.2) by $a_{kn}(t)$ and sum for $k = 1, 2, \ldots\ldots\ldots, n$ so that

$$\langle T_n', T_n \rangle + \langle K T_n, T_n \rangle + \varepsilon \int \left| T_n \right|^5 d\mu = \langle g, T_n \rangle . \tag{3.7}$$

Since

$$\langle T_n', T_n \rangle + \langle T_n', T_n \rangle = \frac{d}{dt}\left(\frac{1}{2} \| T_n \|^2_{L^2(\Omega)} \right),$$

then, integration of (3.7) from 0 to t yields

$$\| T_n(t)\|^2_{L^2(\Omega)} + 2\int_0^t \langle KT_n, T_n\rangle\, dx + 2\varepsilon\int_0^t \| T_n\|^2_{L^5_\mu}\, dx = \| T_n(0)\|_{L^2(\Omega)} + 2\int_0^t \langle g, T_n\rangle\, dx. \quad (3.8)$$

Hence, (3.6) is valid provided that (3.4) holds. Moreover, by Theorem 10 (see Qatanani et al. (2007)) there exists a constant $\widetilde{C}$ such that

$$\begin{aligned} 2\int_0^\tau \langle KT_n, T_n\rangle\, dx &\geq 2\int_0^\tau \tilde{C}\left(\| T_n\|^2_X - 1\right) dx \\ &\geq 2\tilde{C}\| T_n\|^2_{L^2(H^1(\Omega))} - 2\tilde{C}\tau. \end{aligned} \quad (3.9)$$

Upon using the inequality

$$ab < \lambda\,\frac{a^p}{p} + \lambda^{-q/p}\,\frac{b^q}{q},$$
$$\frac{1}{p} + \frac{1}{q} = 1, \quad a, b \geq 0, \lambda > 0,$$

and writing

$$g = g_1 + g_2,\ g_1 \in L^2\left(H^1(\Omega)^*\right), \quad g_2 \in L^{5/4}\left(L^{5/4}_\mu\right),$$

we have

$$\begin{aligned} 2\int_0^\tau \langle g, T_n\rangle\, dx &= 2\int_0^\tau \langle g_1, T_n\rangle_{H^1(\Omega)}\, dx + 2\int_0^\tau \langle g_2, T_n\rangle_\mu\, dx \\ &\leq 2\| T_n\|_{L^2\left(H^1(\Omega)\right)} \| g_1\|_{L^2\left(H^1(\Omega)^*\right)} + 2\| T_n\|_{L^5\left(L^5_\mu\right)} \| g_2\|_{L^{5/4}\left(L^{5/4}_\mu\right)} \\ &\leq \tilde{C}\| T_n\|^2_{L^2\left(H^1(\Omega)\right)} + C(\tilde{C})\| g_1\|^2_{L^2\left(H^1(\Omega)^*\right)} + C(\varepsilon)\| g_2\|^{5/4}_{L^{5/4}\left(L^{5/4}_\mu\right)}. \end{aligned} \quad (3.10)$$

Collecting the estimates (3.8), (3.9) and (3.10) we obtain (3.4). Using the boundedness of *K*, the estimate (3.6) follows immediately from (3.4).

Lemma 3: There exists a solution for the auxiliary problem (2.9)–(2.10).

Proof: For simplicity, we denote the solution by T instead of T_ε. Next, we follow the steps:

1. Existence of solution for the Galerkin equations (3.2)–(3.3). These Galerkin equations can be viewed as a system of ordinary differential equations and hence the existence of solutions can be obtained from the theorem of Caratheodory (see Saldanha and Martins (1991), (1995)). In order to apply this theorem one needs to note the following:

(a) If $\psi(t)$ is a solution of (3.2)–(3.3) then $\|\psi(t)\|_{L^2(\Omega)} \leq C$ by (3.6).

(b) The mapping $t \mapsto \langle K\psi, v_j \rangle$ is measurable on $(0,\tau)$ and for all $\psi \in X_n$.

(c) The mapping $\psi \mapsto \langle K\psi, v_j \rangle$ is continuous on X_n. Let $\{T_n\}$ be a sequence of solutions of (3.2) − (3.3). Then according to Lemma 2 there exist $\psi \in V$, $v \in V^*$ and $y \in L^2(\Omega)$ such that T_n converges weakly to T in V, KT_n converges weakly to v in V^* and $T_n(\tau)$ converges weakly to y in $L^2(\Omega)$ *as* $n \to \infty$.

2. Show that

$$\left(y,\varphi(\tau)\psi\right)-\left(T_0,\varphi(0)\psi\right)=\int_0^\tau \langle g(t)-v(t),\varphi(t)\psi\rangle+\langle \varphi(t)\psi, T(t)\rangle\, dt \qquad (3.11)$$

for all $\varphi \in C^\infty[0,\tau]$ and $\psi \in X$. First, let

$\varphi(t) \in C^\infty[0,\tau]$ and $\psi \in X_m$, $m \leq n$.

Then, integrating by parts yields

$$\begin{aligned}\left(T_n(\tau),\varphi(\tau)\psi\right)-\left(T_n(0),\varphi(0)\psi\right) &= \int_0^\tau \langle T_n'(t), \varphi(t)\psi\rangle + \langle \varphi'(t)\psi, T_n(t)\rangle\, dt \\ &= \int_0^\tau \langle g(t)- KT_n(t),\varphi(t)\psi\rangle + \langle \varphi'(t)\psi, T_n(t)\rangle\, dt \\ &= \langle g - KT_n, \varphi\psi\rangle_V + \langle \varphi'\psi, T_n\rangle_V .\end{aligned}$$

If we let $n \to \infty$ we obtain

$$\begin{aligned}&\langle y, \phi(\tau)\psi\rangle - \left(T_0, \phi(0)\psi\right) \\ &\qquad = \langle g-v, \phi\psi\rangle_V - \langle \phi'\psi, T\rangle_V, \text{ for all } \psi \in X.\end{aligned}$$

3. We prove that T, v and y satisfy

$$T' + v = g \,, \quad T \in W \tag{3.12}$$

$$T(0) = T_0 \,, \quad T(\tau) = y \,. \tag{3.13}$$

From (3.11) it implies that

$$\int_0^\tau \langle g(t) - v(t) \,, \psi \rangle \, \varphi(t) \, dt = -\int_0^\tau \langle \psi, T(t) \, \varphi'(t) \rangle \, dt \tag{3.14}$$

for all $\varphi \in C^\infty[0,\tau]$. This means that T' exists and $T' = g - v \in V^*$. Hence, also $T \in W$. Moreover, the integration by parts yields

$$\left(T(\tau) \,, \phi(\tau)\psi \right) - \left(T(0), \phi(0)\psi \right) = \int_0^\tau \langle T'(t), \, \phi(t)\psi \rangle + \langle \phi'(t)\psi, \, T(t) \rangle \, dt \,, \tag{3.15}$$

for all $\varphi \in C^\infty[0,\tau]$, $\psi \in X$. Hence, (3.11) and (3.15) imply

$$\left(T(\tau) \,, \varphi(\tau)\psi \right) - \left(T(0), \varphi(0)\psi \right) = \left(y, \, \varphi(\tau)\psi \right) - \left(T_0 \,, \varphi(0)\psi \right) \tag{3.16}$$

so that

$$T(\tau) = y \quad \text{and} \quad T(0) = T_0 \,.$$

4. Finally, we prove that $KT = v$. Since $K: V \to V^*$ is pseudomonotone, it satisfies the so–called condition (M); (see Zeidler (1990), Ch.27). This means that the weak convergence of T_n and KT_n together with

$$\overline{\lim_{n \to \infty}} \, \langle \, K \, T_n \,, \, T_n \rangle \;\leq\; \langle \, v \,, \, T \, \rangle \tag{3.17}$$

imply that

$$K \; T \;=\; v \,.$$

Integrating by parts gives

$$\frac{1}{2}\left(\|T_n(\tau)\|^2_{L^2(\Omega)} - \|T_n(0)\|^2_{L^2(\Omega)}\right) = \int_0^\tau \langle T_n'(t), T_n(t)\rangle\, dt = \int_0^\tau \langle g - KT_n, T_n\rangle\, dt$$

or

$$\langle KT_n, T_n\rangle_V = \langle g, T_n\rangle_V + \frac{1}{2}\left(\|T_n(0)\|^2_{L^2(\Omega)} - \|T_n(\tau)\|^2_{L^2(\Omega)}\right).$$

Hence,

$$\overline{\lim_{n\to\infty}}\langle KT_n, T_n\rangle \le \langle g, T\rangle_V + \frac{1}{2}\left(\|T_n(0)\|^2_{L^2(\Omega)} - \|T_n(\tau)\|^2_{L^2(\Omega)}\right)$$

as $T_n(0) \to T(0)$ and

$$\|T(\tau)\|_{L^2(\Omega)} \le \underline{\lim}\|T_n(\tau)\|_{L^2(\Omega)}.$$

This yields (3.17) since,

$$\frac{1}{2}\left(\|T_n(0)\|^2_{L^2(\Omega)} - \|T_n(\tau)\|^2_{L^2(\Omega)}\right) = \int_0^t \langle T', T\rangle\, dt = \langle v - g, T\rangle_V.$$

Consequently, $KT = v$.

4. Existence of Solution for the Original Problem

Lemma 4: Suppose that $T_0 \in L^5(\Omega)$ and assume

$$g(t) \in L^2\big((0,\tau)\times\Omega\big) + L^{5+\delta}\left(L^{5/3+\delta}(\Gamma_1 \cup \Gamma_2)\right) \text{ for } \delta > 0. \quad (4.1)$$

Then, there exists a constant $C > 0$, independent of ε, such that the solution of (2.9)–(2.10) satisfies $\|T_\varepsilon\|_W \le C$.

Proof: It is obvious that T_ε' is bounded in V^* provided that T_ε is bounded in V. Moreover, in virtue of Lemma 2 we can easily see that $\|T_\varepsilon\|_{L^2(X)} \le C$ independent of ε. Hence, it suffices to prove that

$$\int_0^\tau \|T_\varepsilon(t)\|_{L^5_\mu}^5 \, dt \le C\int_0^\tau (\|T_\varepsilon(t)\|_{L^5_{(\omega)}}^5 + \|T_\varepsilon\|_{L^5(\Gamma_2)}^5)\, dt$$

is bounded uniformly in ε. In fact we will prove a stronger statement

$$\|T_\varepsilon\|_{L^{25/3}((0,\tau)\times\Omega)} + \|T_\varepsilon\|_{L^5(L^{10}(\Gamma_1\cup\Gamma_2))} \le C\left(g, T_0, \|T\|_{L^2(X)}\right). \tag{4.2}$$

We use a Moser–type argument to derive abound for $(T_\varepsilon)^+$. The proof for $(T_\varepsilon)^-$ is similar. For simplicity we denote T_ε by T and define $B: R_+ \to R_+$ by $B(t) = t^\alpha$, $\alpha \ge 1$. Then, set

$$T_k = \min\left\{T^+, k\right\}, \ k > 0$$

and define

$$f(T_k) = \int_0^{T_k} (B'(t))^2 \, dt = \frac{\alpha^2}{2\alpha - 1} T_k^{2\alpha-1}, \ f(T_k) \in X^+.$$

Next, suppose that T is a solution of (2.9)–(2.10), i.e. $T(0) = T_0$ and

$$\langle T'(t), \psi \rangle + \langle K T(t), \psi \rangle + \varepsilon \int |T_n(t)|^3 T(t) \psi \, d\mu \le \langle g(t), \psi \rangle$$

for all $\psi \ge 0$, $\psi \in X$.

Then, choosing $\psi = f(T_k)$ we have

$$\langle T', f(T_k) \rangle + C\|B(T_k)\|_{L^6(\Omega)}^2 + C\|B(T_k)\|_{L^4(\Gamma_1)}^2 \le \langle g, f(T_k)\rangle + C\|B(T_k)\|_{L^2(\Omega)}^2.$$

Next, integrating from 0 to t and let $k \to \infty$, then

$$\int_0^t \langle g, f(T_k)\rangle + \|B(T_k)\|_{L^2(\Omega)}^2 dt$$

$$\to C(\alpha)\left(\int_0^t\int_\Omega (g+T_+)T_+^{2\alpha-1}\,dz\,dt+\int_0^t\int_{\Gamma_1\cup\Gamma_2} g\,T_+^{2\alpha-1}\,dz\,dt\right) \tag{4.3}$$

and

$$\int_0^t \langle T',\, f(T_k)\rangle\, dt \to \int_0^t \langle T',\, f(T_+)\rangle\, dt$$

$$= C(\alpha)\int_0^t\int_\Omega \frac{d}{dt}u_+^{2\alpha}\,dz\,dt \tag{4.4}$$

$$= C(\alpha)\left(\left\| T_+^\alpha(t)\right\|_{L^2(\Omega)}^2 - \left\| (T_0)_+^\alpha\right\|_{L^2(\Omega)}^2\right).$$

Hence, collecting all estimates we obtain

$$\left\| T_+^\alpha \right\|_{L^\infty\left(L^2(\Omega)\right)}^2 + \left\| T_+^\alpha \right\|_{L^2\left(L^6(\Omega)\right)}^2 + \left\| T_+^\alpha \right\|_{L^2\left(L^4(\Gamma_1\cup\Gamma_2)\right)}$$

$$\le C(\alpha)\left\{C_1\left\| T_+^{2\alpha-1}\right\|_{L^{q_1}\left(L^{q_2}(\Omega)\right)} + C_2\left\| T_+^{2\alpha-1}\right\|_{L^{q_3}\left(L^{q_4}(\Gamma_1\cup\Gamma_2)\right)} + C_3\right\}, \tag{4.5}$$

where

$$C_1 = \left\| g+T_+\right\|_{L^{p_1}\left(L^{p_2}(\Omega)\right)},\ C_2 = \left\| g\right\|_{L^{p_3}\left(L^{p_4}(\Gamma_1\cup\Gamma_2)\right)},\ C_3 = \left\| (T_0)_+^\alpha\right\|_{L^2(\Omega)}^2$$

and the pairs of conjugate exponents p_i, q_i, i = 1, 2, 3, 4 are to be determined. Interpolating between the norms we obtain

$$\left\|\psi\right\|_{L^{10/3}((0,\tau)\times\Omega)} \le C\left(\left\|\psi\right\|_{L^\infty\left(L^2(\Omega)\right)} + \left\|\psi\right\|_{L^2\left(L^6(\Omega)\right)}\right),$$

for all $\psi \in L^\infty\left(L^2(\Omega)\right)\cap L^2\left(L^2(\Omega)\right)$ (see for example Laitinen and Tiihonen (2001)). Finally, the conclusion follows by iteration of (4.5) if the constants C_1, C_2 and C_3 are finite and the powers of T^+ are greater on the left–hand side of (4.5) than on the right–hand side for all $1 \le \alpha \le 5/2$. Namely we require $10/3 > q_1(2\alpha-1)$, $10/3 > q_2(2\alpha-1)$, $2 > q_3(2\alpha-1)$, $4 > q_4(2\alpha-1)$. In fact p_1, p_2, p_3 and q_1, q_2, q_3 can be determined if T_0 and g have assumed regularity. Moreover, the terms on the right–hand side of (4.5) can be bounded by $\| T \|_{L^2(X)}$ when $\alpha = 1$ and hence the bounding constant in (4.3) does not depend on ε.

Theorem 2: Suppose that the hypotheses of Lemma 4 hold, then there exists a solution for (2.7)–(2.8).

Proof: From the priori estimate of Lemma 4, it follows that the sequence $\{T_k\}$ is bounded in *W*. Hence, there is $T \in W$ such that T_ε converges weakly to T in W as $\varepsilon \to 0$. By repeating the arguments of Lemma 3 one sees easily that T solves (2.7) – (2.8).

5. Uniqueness of the Solution

Theorem 3: Assume that T_1 and T_2 are solutions of (2.7)–(2.8) corresponding to the right–hand sides $g_1, g_2 \in V^*$ and the initial data $T_0^1, T_0^2 \in L^2(\Omega)$ satisfying

$$\langle g_1 - g_2, v \rangle_V \geq 0, \qquad \forall\, v \geq 0, v \in W$$

$$\left(T_0^1 - T_0^2, \psi \right) \geq 0, \qquad \forall\, \psi \geq 0, \psi \in L^2(\Omega).$$

Then, $T_1 \geq T_2$ almost everywhere in $\Omega \times (0, \tau)$ with respect to the measures $L^1 \times L^n$, $L^1 \times \mu$ and $L^1 \times \gamma$. Consequently, if the solution of (2.7)–(2.8) exists, then it is unique.

Proof: To prove the uniqueness of the solution we follow the same approach used for the stationary case in (see Qatanani et al. (2007)). The only difference is instead of Ω_0 defined in the proof of Theorem 11 (see Qatanani et al. (2007)), we consider the following set

$$\left\{ (z,t) \in \overline{\Omega} \times [0,\tau] : T_1(z,t) < T_2(z,t) \right\},$$

which is of measure zero. For any $t \in [0,\tau]$ we define $\Omega_0(t), \Omega_\varepsilon(t)$ and $\psi_\varepsilon(z,t)$ as in (see Qatanani et al. (2007)). Our goal is to prove that

$$\int_0^\tau \mu\big(\Omega_\varepsilon(t)\big)\,dt = \int_0^\tau \int_{\Omega_\varepsilon(t)} d\mu\,dt = \int_0^\tau \varepsilon^{-1}\left(\int_{\Omega_\varepsilon(t)} \varepsilon^5\,d\mu\right)^{1/5} dt$$

$$\le \varepsilon^{-1}\int_0^\tau \|\psi_\varepsilon\|_{L^5_\mu}\,dt \le \varepsilon^{-1}C(\tau)\|\psi_\varepsilon\|_{L^2(L^5_\mu)} \to 0.$$

This is done by showing that

$$\int_0^\tau \|\psi_\varepsilon\|^2_{L^2_\mu}\,dt \le C\int_0^\tau \int_\Omega a_{ij}\,\partial_i\psi_\varepsilon\,\partial_j\psi_\varepsilon\,dz\,dt$$

$$+C\int_0^\tau \left(\int_{\Gamma_2} \xi\|\psi_\varepsilon\|^{p+1}\,dx\right)^{2/p+1} + C\int_0^\tau \left(\int_{\Gamma_1} G\,\psi_\varepsilon^4\,\psi_\varepsilon\,d\mu\right)^{2/5} dt$$

and

$$\int_0^\tau \int_\Omega a_{ij}\,\partial_i\psi_\varepsilon\,\partial_j\psi_\varepsilon\,dz\,dt \le \varepsilon\|\psi_\varepsilon\|_{L^2(L^5_\mu)}\,g_\varepsilon - f_\varepsilon \tag{5.1}$$

$$\int_0^\tau \left(\int_\Gamma \xi|\psi_\varepsilon|^{p+1}\,dx\right)^{2/p+1} + \left(\int_{\underline{\Omega}} G\,\psi_\varepsilon^4\,\psi_\varepsilon\,d\mu\right)^{2/5} dt$$

$$\le \varepsilon\|\psi_\varepsilon\|_{L^2(L^5_\mu)}\,g_\varepsilon + h_\varepsilon, \tag{5.2}$$

where $g_\varepsilon \to 0$ and $h_\varepsilon - f_\varepsilon$ can be neglected for sufficiently small ε. The derivation of the estimates (5.1) and (5.2) is done as in (see Qatanani et al. (2007)) except for adding integration over $(0,\tau)$ to all terms. The main difference is the appearance of the time derivative when deriving (5.1). However, this additional term can be treated in the following manner:

First integrating by parts we obtain

$$-\int_0^{\tau} \langle\, T_2'(t) - T_n'(t)\,,\, \psi_{\varepsilon}(T)\,\rangle\, dt$$

$$= \int_0^{\tau} \langle\, T_2 - T_1\,,\, \psi_{\varepsilon}'\,\rangle\, dt - \big(\, T_2(\tau) - T_1(\tau)\,,\psi_{\varepsilon}(T)\big).$$

Since $T_2 - T_1 = \psi_{\varepsilon}$ in $\Omega_0(t)\backslash\Omega_{\varepsilon}(t)$ and $\psi_{\varepsilon}'(t) = 0$ in the complement of $\Omega_0(t)\backslash\Omega_{\varepsilon}(t)$, we have

$$\int_0^{\tau} \langle\, T_2 - T_1\,,\, \psi_{\varepsilon}'\,\rangle\, dt = \int_0^{\tau} \langle \psi_{\varepsilon}\ \ ,\psi_{\varepsilon}'\,\rangle\, dt = \frac{1}{2}\,\big(\,\psi_{\varepsilon}(\tau), \psi_{\varepsilon}(\tau)\,\big),$$

and therefore

$$-\int_0^{\tau} \langle\, T_2' - T_1', \psi_{\varepsilon}\,\rangle\, dt = \left(\ \psi_{\varepsilon}(\tau), \frac{1}{2}\psi_{\varepsilon}(\tau) - \big(T_2(\tau) - T_1(\tau)\big)\right) \le 0\,.$$

Hence, the additional term can be ignored and the proof of uniqueness can be completed as in the proof of Theorem 11 (see Qatanani et al. (2007)).

6. Conclusion

In this article, we proved the existence and the uniqueness for the time–dependent conductive–radiative problem. We have restricted ourselves to grey materials, that is, the radiative coefficients of these materials do not depend on a wavelength. Also temperature dependent material properties are beyond the scope of this work. The results presented in this work could be generalized to some cases involving materials with wavelength on temperature dependent radiative properties, but this part of the theory will be treated in later works. The mathematical analysis of non–grey models is entirely an open problem.

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