An- Najah J. Res. , Vol . III, 9. (1995) Mahoud A. Masri

ON COMPOSITION OPERATORS ON A^2

BY
MAHMUD ILAYYAN MASRI
DEPARTMENT OF MATHEMATICS
AN-NAJAH NATIONAL UNIVERSITY

ABSTRACT

If Φ is an analytic function mapping the open unit disk D into itself and A^2 is the Bergman space of analytic functions on D, the compositon operator C_Φ on A^2 is defined by $C_\Phi f = f o \Phi \forall f \varepsilon A^2$.

In this paper we consider the spectral radius, unitary equivalence, subnormality of C_Φ and study the case $\Phi(z) = z^m$, m = 2, 3, ... in detail.

ملخص

اذا كانت Φ داله تحليليه من القرص المفتوح D الى نفسه و A^2 هو فراغ برغمان المكون من الدوال التحليليه على D والتي تحقق شرط تكامل معين فانه يمكن تعريف المؤثر المركب C_Φ على A^2 كما يلي: $C_\Phi f = f \circ \phi$ لكل f تنتمي الى A^2.

في ورقة البحث هذه ندرس نصف القطر الطيفي والتكافؤ الأحادي والصفه شبه الطبيعيه للمؤثر المذكور أعلاه. وكذلك نبحث بشيء من التفصيل في الحاله الخاصه

$\Phi(z) = z^m$, m = 2, 3, ...

Hence, the norm of f is given by

$$\|f\|^2 = \langle f, f\rangle = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$$

Of special interest is the function $K_\zeta(z) = (1-\zeta z)^{-2}$ which serves as the "reproducing kernel" for A^2, i.e.,

$$f(\zeta) = \langle f, k_\zeta\rangle \forall\ f \in A^2 \ \&\ \ \forall \zeta \in D.$$

Furthermore, the functions $e_n(z) = \sqrt{n+1}\ z^n$, $n = 0,1,2,\ldots$ form an orthonormal basis for A^2.
If Φ is a non-constant analytic function mapping D into itself, then Φ induces a composition operator $C_\Phi : A^2 \to A^2$ defined by $C_\Phi f = f \circ \Phi\ \forall\ f \in A^2$.
Boyd [1] showed that C_Φ is bounded and obtained norm estimats for C_Φ. He studied normal, unitary, hermetian and compact composition operators on A^2. Furthermore, he computed the spectrum of C_Φ for some special kinds of Φ.
Cowen [6] computed the spectral radius of C_Φ as an operator on the Hardy space H^2. Here, we compute the spectral radius of C_Φ as an operator on A^2. Also, in recent work Campbell-Wright [2] found a necessary and sufficient condition for two composition operators on H^2 to be unitarily equivalent.We show that the same thing holds in the case of A^2. Moreover, we give a necessary condition for the subnormality of C_Φ on A^2.

Finally, as an example we study C_Φ when $\Phi(z)= z^m$, m = 2,3,...

2.Spectral raduis. It was found out that the fixed points of Φ are related to some properities of C_Φ and to its spectral radius in particular. We say that a point $b\in\overline{D}$, the closure of D, is a fixed point of Φ if $\text{limit}_{r\to 1^-}\Phi(rb) = b$. We write $\text{limit}_{r\to 1^-}\Phi'(rb) = \Phi'(b)$. Although it is not apriori evident that Φ has fixed points the following is known.

Denjoy-Wolff Theorem [8,9] : let $\Phi : D \to D$ be analytic and non-elliptic Mobius transformation onto D. Then $\exists$ a unique fixed point a of Φ in $\overline{D}$ such that $|\Phi'(a)| \leq 1$.

We call the distinguished fixed point a the Denjoy-Wolff point of Φ and we point out that if $|a| = 1$, then $0 < \Phi'(a) \leq 1$ and if $|a|<1$, then $0 \leq |\Phi'(a)| < 1$. Now we are ready to prove the spectral radius theorem which is similar to that of [6] in the H^2 case.

Spectral radius theorem : Let $\Phi : D \to D$ be analytic with Denjoy-Wolff Point a. Then the spectral radius $r(C_\Phi)$ of C_Φ is 1 when $|a| < 1$ and $(\Phi'(a))^{-1}$ when $|a| = 1$.

Proof : $r(C_\Phi) = \text{limit}_{n\to\infty} \| C^n_\Phi\|^{1/n}$
$= \text{limit}_{n\to\infty} \|C_{\Phi n}\|^{1/n}$

Where $\Phi_n = \Phi 0 \Phi_{n-1}$, n=1,2,... $\Phi_1 = \Phi$ and $\Phi_0(z) = z$ $\forall$ $z \in D$ (see e.g., [4,p.142]).
Boyd [1] showed that

$$(1-|\phi(0)|^2)^{-1} \le \|C_\phi\| \le \frac{1+|\phi(0)|}{1-|\phi(0)|}$$

Hence,

$$\lim_{n\to\infty} sup\ (1+|\phi_n(0)|^2)^{-1/n} \le r(C_\phi) \le$$
$$\lim_{n\to\infty} inf\ \left(\frac{1+|\phi_n(0)|}{1-|\phi_n(0)|}\right)^{1/n}$$

Since,

$$\lim_{n\to\infty} inf\ \left(\frac{1+|\phi_n(0)|}{1-|\phi_n(0)|}\right)^{1/n} =$$
$$\lim_{n\to\infty} inf\ (1+|\phi_n(0)|)^{2/n}(1-|\phi_n(0)|^2)^{-1/n}$$
$$= \lim_{n\to\infty} inf\ (1-|\phi_n(0)|^2)^{-1/n}$$

We have

$$r(C_\phi) = limit_{n\to\infty}(1-|\phi_n(0)|^2)^{-1/n}$$

$$= limit_{n\to\infty}(1-|\phi_n(0)|)^{-1/n} \qquad (2.1)$$

Since (see [5]) $\text{limit}_{n\to\infty}\ \Phi_n(0) = a$, $r(C_\Phi) = 1$ if $|a| < 1$ by (2.1). When $|a| = 1$ and $\Phi'(a) < 1$ we have [3,p.32]

$$limit_{n\to\infty}\frac{1-|\phi_n(0)|}{1-|\phi_{n-1}(0)|} = \phi'(a)$$

Therefore, (2.1) implies

$$r(C_\phi) = limit_{n\to\infty}(1-|\phi_n(0)|)^{-1/n}$$

$$= limit_{n\to\infty}\left(\Pi_{k=0}^{n-1}\frac{1-|\phi_k(0)|}{1-|\phi_{k+1}(0)|}\right)^{1/n}$$

$$= limit_{n\to\infty}\frac{1-|\phi_{n-1}(0)|}{1-|\phi_n(0)|} = (\phi'(a))^{-1}$$

Next, suppose that $|a| = 1$ and $\Phi'(a) = 1$. If $\{z_n\}$ is a sequence in D converging to a such that $\Phi(z_n) \to a$ as $n \to \infty$ and

$$\alpha = limit_{n\to\infty} \frac{1-|\phi(z_n)|}{1-|z_n|}$$

exists then by [3,pp25-32] $\alpha \geq \Phi'(a) = 1$. Hence , letting
$z_n = \Phi_{n-1}(0)$, $n = 1,2, \ldots$, we get

$$\lim_{n\to\infty} inf \frac{1-|\phi_n(0)|}{1-|\phi_{n-1}(0)|} \geq 1.$$

Therefore, by (2.1)

$$r(C_\phi) = limit_{n\to\infty} (\Pi_{k=0}^{n-1} \frac{1-|\phi_k(0)|}{1-|\phi_{k+1}(0)|})^{1/n}$$

$$\leq \lim_{n\to\infty} sup \frac{1-|\phi_{n-1}(0)|}{1-|\phi_n(0)|} \leq 1$$

But (2.1) again implies $r(C_\Phi) \geq 1$ since $1-|\Phi_n(0)| \leq 1$ $\forall$ n = 1,2,... . Thus $r(C_\Phi) = 1 = (\Phi'(a))^{-1}$.

3. Unitary equivalence. Campbell-Wright [2] proved a theorem concerning unitary equivalence of composition operators on H^2. Here we show that the same thing holds in the A^2 case.

Theorem : Let Φ and Ψ be analytic functions, not disk automorphisms, that map D into itself. Suppose that the Denjoy-Wolff point a of Φ is in D with $\Phi_n(0) \neq a$ $\forall$ positive integers n . Then C_Φ is unitarily equivalent to C_Ψ on A^2 iff $\Psi(z) = e^{i\theta}$ $\Phi(e^{-i\theta} z)$ for some real number θ.

Proof : Let U be a unitary operator on A^2 such that $C_\Psi = U^* C_\Phi U$. Since $|a| < 1$ we have $0 \leq |\Phi'(a)| < 1$. Thus [5] implies that the non-zero solutions of the equation $f o \Phi = f$ are the constant functions. Hence, the same is true for the equation $f o \Psi = f$ by the unitary equivalence of C_Φ and C_Ψ. Therefore, $U(1) = \gamma\, 1$ where $|\gamma| = 1$. Since $K_0 = 1$ and $C^*_\Phi k_\alpha = K_{\Phi(\alpha)}$ where C^*_Φ is the adjoint of C_Φ, it follows that $\forall$ n

$$UK_{\Psi_n(0)} - UC^*_{\Psi_n}(k_0) - UC^{*^n}_{\Psi}(k_0) - C^{*^n}_{\Phi} U(K_0) - \gamma C^{*^n}_{\Phi}(k_0) - \gamma K_{\Phi_n(0)}$$

In particular, when n = 1, we get

$$(1-|\psi(0)|^2)^{-1}=\|k_{\psi(0)}\|=\|Uk_{\psi(0)}\|=\|\gamma k_{\phi(0)}\|$$

$$=\|k_{\phi(0)}\|=(1-|\phi(0)|^2)^{-1}$$

Therefore, $\Phi(0) = e^{-i\theta}\,\Psi(0)$ for some real number θ. Furthermore,

$$(1-\overline{\phi}(0)\phi_n(0))^{-2}=k_{\phi(0)}(\phi_n(0))=\langle k_{\phi(0)},k_{\phi_n(0)}\rangle$$

$$=\langle\gamma k_{\phi(0)},\gamma k_{\psi_n(0)}\rangle=\langle Uk_{\psi(0)},Uk_{\psi_n(0)}\rangle$$

$$=\langle k_{\psi(0)},k_{\psi_n(0)}\rangle = k_{\psi(0)}(\psi_n(0)$$

$$=(1-\overline{\psi}(0)\psi_n(0))^{-2}$$

Thus, $\Phi_n(0) = (\overline{\Psi(0)}/\overline{\Phi(0)})\ \Psi_n(0) = e^{-i\theta}\ \Psi_n(0)$. It follows that the analytic functions $\Psi(z)$ and $e^{i\theta}$ $\Phi(e^{-i\theta}z)$ agree on the sequence $\{e^{i\theta}\ \Phi_n(0)\}$ which converges to $e^{i\theta}$ a in D and hence $\Psi(z) = e^{i\theta}\Phi(e^{-i\theta}$ $z)$.

Conversely, if $\beta(z) = e^{i\theta}$ z , $z \in D$, then by [1] C_β is a unitary operator on A^2 and C_Ψ = $C^*{}_\beta C_\Phi C_\beta$.

4. Subnormality of C_Φ on A^2 . Boyd [1] proved that C_Φ is normal on A^2 iff $\Phi(z) = \alpha z$ for some α with $|\alpha|\leq 1$ iff $C^*{}_\Phi$ is a composition operator. Here, we give a necessary condition for the subnormality of C_Φ on A^2. Let S be an operator on a Hilbert space

H. S is called subnormal if there is a Hilbert space K containing H and a normal operator N on K such that N leaves H invariant and S is the restriction of N to H. Also, S is called hyponormal if $S^* S \geq S S^*$ where S^* is the adjoint of S.

Theorem 4.1 : If $\exists$ a positive integer n such that

$$\|C_\phi^2 e_n\| < \|C_\phi e_n\|^2 \qquad (4.1)$$

then C_Φ is not subnormal.

Proof : Suppose $\exists$ n as in (4.1). Let $f_o = \beta e_n$ and $f_1 = \gamma e_n$ where β and $\gamma \in R$. It follows that

$$\sum_{j,k=0}^{1} \langle C_\phi^{j+k} f_j, C_\phi^{j+k} e_k \rangle =$$

$$\langle f_\circ, f_\circ \rangle + \langle C_\phi f_1, C_\phi f_\circ \rangle + \langle C_\phi f_\circ, C_\phi f_1 \rangle + \langle C_\phi^2 f_1, C_\phi^2 f_1 \rangle$$

$$= \beta^2 + 2\beta\gamma \|C_\phi e_n\|^2 + \gamma^2 \|C_\phi^2 e_n\|$$

$$= g(\beta, \gamma)$$

Hence, (4.1) implies that the function $g(\beta,\gamma)$ has a saddle point at (0,0). Thus $\exists$ non-zero $\beta,\gamma \in \mathbf{R}$ such that

$$\sum_{j,k=0}^{1} \langle C_\Phi^{k+j} f_j, C_\Phi^{j+k} f_k \rangle < 0.$$

Therefore, [4,p.117] implies that C_Φ is not subnormal.

In [7] Cowen and Kriete proved that $\Phi(0) = 0$ if C_Φ is hyponormal on H^2. We conjecture that the same result is true for A^2. Moreover, the next results are similar to theirs.

Lemma : If $0 < |a| < 1$ or if $|a| = 1$ and $\Phi'(a) = 1$, then neither C_Φ nor C^*_Φ is hyponormal on A^2.

Proof : The spectral radius theorem implies $r(C_\Phi)=1$ but $\|C_\Phi\| > 1$. Therefore, [4,p.141] implies that neither C_Φ nor C^*_Φ is hyponormal on A^2.

We note that in the lemma neither C_Φ nor C^*_Φ is subnormal on A^2 since subnormality implies hyponormality[4,p. 140].

Theorem 4.2 : If C^*_Φ is hyponormal on A^2, then $|a| = 1$ and $\Phi'(a) < 1$, or else C_Φ is normal on A^2.

Proof : By the lemma we need only examine the case $\Phi(0) = 0$.
We have $S = z^k A^2$ is an invariant subspace of C_Φ on

A^2 ∀ positive integers k. Hence , $S^\perp$ is an invariant subspace of C^*_Φ .Sine $S^\perp$ is finite dimensional and C^*_Φ hyponormal on it, [4,p. 142] implies that C^*_Φ is normal on $S^\perp$.
Therefore, by [1] $\Phi(z) = \alpha z$ for some α with $|\alpha| \leq 1$ and consequently C_Φ is normal.

5. Example . Let $\Phi(z) = z^m$, $m = 2,3,\ldots$ and C_Φ be the induced composition operator on A^2. We prove that

a) $\sigma(C_\Phi) = \{\lambda \in C : |\lambda| \leq 1/\sqrt{m}\} \cup \{1\}$ (5.1)

Where $\sigma(C_\Phi)$ is the spectrum of C_Φ.

b) C_Φ is bounded below by $1/\sqrt{m}$

c) C_Φ is not subnormal.

Proof : a) Let $f(z) = \Sigma^\infty_{k=0} a_k z^k$ and $g(z) \neq bz$ where $b = 0$.
Suppose $\lambda \neq 1$ and $(C_\Phi - \lambda I)(f) = g$. Then

$$\sum_{k=0}^{\infty} a_k z^{mk} = bz + \sum_{j=0}^{\infty} \lambda a_j z^j$$

Fixing m and equating the corresponding coefficients we get

$$a_1 = -\frac{b}{\lambda} \quad and \quad a_{mk} = \frac{a_k}{\lambda}, \quad k = 1, 2, 3, \ldots$$

Hence,

$$a_{m^n} = -\frac{b}{\lambda^{n+1}}, n = 1, 2, 3, \ldots$$

Thus,

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1} \geq \sum_{n=1}^{\infty} \frac{|a_{m^n}|^2}{m^n+1} = |b|^2 \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n+2}(m^n+1)}$$

Therefore, the ratio test implies there does not exist $f \in A^2$ such that $(C_\Phi - \lambda I)(f) = g$ if $|\lambda| < 1/\sqrt{m}$ which means

$$\{\lambda \in \mathbb{C} : |\lambda| < 1/\sqrt{m}\} \subseteq \sigma(C_\Phi) \qquad (5.2)$$

Next let $A^2_0 = \{ f \in A^2 : f(0) = 0 \}$. If $C_\Phi|_{A^2_0}$ is the restriction of C_Φ to A^2_0 and $f(z) = \Sigma^\infty_{k=1} a_k z^k \in A^2$, then for each n = 1,2,... we have

$$\|(C_\phi|_{A_0^2})^n(f)\|^2 = \|f \circ \phi_n\|^2 = \|\sum_{k=1}^{\infty} a_k z^{m^n k}\|^2$$

$$= \sum_{k=1}^{\infty} \frac{|a_k|^2}{(k+1)} \frac{(k+1)}{(m^n k+1)}$$

Since $(k+1)/(m^n k+1)$ decreases to $1/m^n$ as $k \to \infty$ we get

$$(1/m^n)\|f\|^2 \le \|(C_\phi|_{A_0^2})^n(f)\|^2 \le (2/(m^n+1))\|f\|^2$$

Therefore,

$$(1/\sqrt{m}) \le \|(C_\phi|_{A_0^2})^n\|^{1/n} \le (2/m^n+1)^{1/2n}$$

letting $n \to \infty$ it follows that

$$r(C_\phi|_{A_0^2}) = 1/\sqrt{m} \qquad (5.3)$$

Next if $C_\Phi|_C$ is the restriction of C_Φ to the complex numbers then by [5] the only non-zero solutions of $(C_\Phi - \lambda I)(f) = 0$ is $\lambda = 1$ and f constant. So if $\lambda \neq 1$, then the kernel of $C_\Phi - \lambda I$ is zero. Moreover, $\forall$ constant α

$$(C_\phi|_C - \lambda I)\left(\frac{\alpha}{1-\lambda}\right) = \alpha$$

i.e., $C_\Phi|_C - \lambda I$ is onto and hence invertible. Therefore,

$\sigma(C_\Phi|_C) = \{1\}$.

Finally , since

$$\sigma(C_\phi) = \sigma(C_\phi|_{A_0^2} + C_\phi|_C) = \sigma(C_\phi|A_0^2) \bigcup \sigma(C_\phi|_C)$$

(see e.g., [4,p. 43]) and observing that 1 is an eigenvalue of C_Φ (5.2) and (5.3) imply (5.1).

(b) let $f(z) = \Sigma^\infty_{K=0} a_k z^k \in A^2$. Since $(k+1)/(mk+1)$ decreases to $1/m$ as $k \to \infty$ we see that C_Φ is bounded below by $1/\sqrt{m}$ from

$$\|C_\phi f\|^2 = \|\sum_{k=0}^\infty a_k z^{mk}\|^2 = \sum_{k=0}^\infty \frac{|a_k|^2}{(k+1)}\left(\frac{k+1}{mk+1}\right) \geq \frac{1}{m}\|f\|^2$$

(c) Theorem 4.1 implies that C_Φ is not subnormal because

$$\|C_\phi^2 e_k\| = \sqrt{\frac{2}{m^2+1}} < \frac{2}{m+1} = \|C_\phi e_1\|^2$$

We close this example by pointing out that

$$C_\phi e_k = \sqrt{\frac{1+k}{1+mk}}\, e_{mk}\ , \quad k = 0, 1, 2, \ldots$$

and

$$C_\phi^* e_k = \begin{cases} \sqrt{\dfrac{1+(k/m)}{1+k}}\, e_{k/m} & if\ (k/m) \in \mathbb{N} \\ 0 & if\ (k/m) \notin \mathbb{N} \end{cases}$$

Where N is the natural numbers.

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