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INFLUENCE OF EXTERNAL FIELDS ON THE KILLINGBECK POTENTIAL: QUASI EXACT SOLUTION

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The Killingbeck potential consists of oscillator potential plus Cornell potential, i.e. $ar^2 + br - c/r$, that it has received a great deal of attention in particle physics. In this paper, we study the energy levels and wave function for arbitrary m-state in two-dimensional (2D) Schrödinger equation (SE) with a Killingbeck potential under the influence of strong external uniform magnetic and Aharonov–Bohm (AB) flux fields perpendicular to the plane where the interacting particles are confined. We use the wave function Ansatz method to solve the radial problem of the Schrödinger equation with Killingbeck potential. We obtain the energy levels in the absence of external fields and also find the energy levels of the familiar Coulomb and Harmonic oscillator potentials.

Keywords: Schrödinger equation; external magnetic and Aharonov–Bohm flux fields; Killingbeck potential; wave function Ansatz; harmonic and Coulomb potentials.

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1. Introduction

Solutions of fundamental dynamical equations are of great interest in many fields of physics and chemistry. The exact solutions of the SE are possible only for a few potentials. In this regards, the exact solutions of the SE for a hydrogen atom (Coulombic) and a harmonic oscillator represent two typical examples in quantum mechanics.[1–5] SE has been solved with different potentials and methods.[6–17]

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The Killingbeck potential[18,19] consist of harmonic oscillator plus Cornell potential, i.e.

$$V(r,\phi) = ar^2 + br - \frac{c}{r}\,, \tag{1}$$

where a, b and c are free parameters. Due to its importance, it has been employed extensively in condensed matter, atomic, and molecular physics, and has an useful application in the quarkonium spectroscopy and in a hydrogen atom to describe the Stark effect.[20−28] It may be stressed that a specific situation can emerge when the parameters a and b are simultaneously small, and leads to a particular atomic description of a perturbed Coulomb problem.[26] This potential was investigated by applying power series expansion and the supersymmetric perturbation theory.[29,30] Boumedjane *et al.* investigated the lowest energy states and the corresponding wave functions for the generalized Killingbeck potential within the context of the recently introduced differential quadratic method.[31] Aygun *et al.* presented an alternative approach, the asymptotic iteration method, to solve the two-dimensional (2D) radial Schrödinger equation for the Killingbeck potential in a magnetic field.[32] Recently, Hamzavi *et al.* have studied the Dirac equation for the Killingbeck potential to obtain the energy eigenvalues and the corresponding wave functions in the presence of spin and pseudo-spin symmetries by using wave function Ansatz method.[33,34]

The motivation of the present work is to study the nonrelativistic energy eigenvalues and wave functions for a particle in the Killingbeck (harmonic plus Cornell) field and subjected to external uniform magnetic and Aharonov–Bohm (AB) flux fields following the same method used previously in Ref. 33. So, we solve the SE equation for the Killingbeck potential in 2D space in the presence of external magnetic and AB flux fields by using wave function Ansatz method.[35−37]

The structure of this paper is as follows. In Sec. 2, we introduce the SE in the presence of external fields. Then, the resulting radial SE is applied to deal with the Killingbeck potential in order to obtain the energy eigenvalue formula and the corresponding wave function. Section 3 is devoted for discussions of some particular cases from our solution including the cases of harmonic oscillator and Coulomb potentials. We end with summary and conclusions in Sec. 4.

2. Radial Schrödinger Equation for Killingbeck Potential in the External Fields

We consider a 2D charged particles with charge, e, and effective mass, μ, interacting via a radially symmetrical interaction potential $V(r,\phi)$ under the influence of external uniform magnetic field, $\mathbf{B} = B\hat{z}$ and an AB flux field, applied simultaneously. The SE with a vector potential $\mathbf{A}$ and repulsive interaction potential $V(r,\phi)$ can be written as[1,2,38]

$$\left[\frac{1}{2\mu}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + V(r,\phi)\right]\psi(r,\phi) = E\psi(r,\phi)\,, \tag{2}$$

where the vector potential $\mathbf{A}$ may be represented as a sum of two terms, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ such that $\boldsymbol{\nabla} \times \mathbf{A}_1 = \mathbf{B}$ and $\boldsymbol{\nabla} \times \mathbf{A}_2 = 0$ where $\mathbf{B} = B\hat{z}$ is the applied magnetic field, and $\mathbf{A}_2$ describes the additional magnetic flux $\Phi_{\rm AB}$ created by a solenoid. Hence, the vector potentials have the following azimuthal components[39–42]

$$\mathbf{A}_1 = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \frac{Br}{2}\hat{\phi}, \quad \mathbf{A}_2 = \frac{\Phi_{\rm AB}}{2\pi r}\hat{\phi}, \quad \mathbf{A} = \left(\frac{Br}{2} + \frac{\Phi_{\rm AB}}{2\pi r}\right)\hat{\phi}\,. \tag{3}$$

We use the 2D cylindrical wave function as

$$\psi(r,\phi) = \frac{1}{\sqrt{2\pi}}\frac{R(r)}{\sqrt{r}}e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots\,, \tag{4}$$

where m is the magnetic quantum number. Substituting the wave function (4) and Killingbeck potential (1) into the SE (2), we obtain a second order differential equation satisfying the radial part of the wave function; namely $R(r)$ in units $\hbar = c = 1$,

$$\frac{d^2R(r)}{dr^2} + \left(\varepsilon - \gamma^2 r^2 - b_1 r + \frac{c_1}{r} - \frac{m'^2 - 1/4}{r^2}\right)R(r) = 0\,, \tag{5}$$

where

$$\varepsilon = 2\mu E - \mu\omega_c m', \quad \gamma^2 = \left(\frac{\mu\omega_c}{2}\right)^2 + 2\mu a, \quad b_1 = 2\mu b, \quad c_1 = 2\mu c\,, \tag{6}$$

and

$$m' = m + \xi, \quad \xi = \frac{\Phi_{\rm AB}}{\Phi_0}, \quad \Phi_0 = \frac{2\pi}{e}, \quad \omega_c = \frac{eB}{\mu}\,. \tag{7}$$

It is worth noting that ξ is taken as an integer with the flux quantum Φ_0 and ω_c is the cyclotron frequency. The AB flux field inside solenoid adds a new quantum number in the present model.

In developing a solution to Eq. (5), we start with the following choice of the wave function in the form

$$R_{nm}(r) = f_n(r)\exp[g_m(r)]\,, \tag{8}$$

where the functions $f_n(r)$ in above equation assigns the number of nodes to the wave function and is defined by[33–37]

$$f_n(r) = \begin{cases} 1 & \text{for } n = 0\,, \\ \displaystyle\prod_{i=1}^{n}(r - \alpha_i^n) & \text{for } n \geq 1\,, \end{cases} \tag{9}$$

Further, the function $g_m(r)$ should be in the form

$$g_m(r) = -\frac{1}{2}pr^2 - qr + \delta\ln r\,, \tag{10}$$

with p, q and δ as positive parameters whose values are to be determined in terms of potential parameters a, b and c. The wave function (8) satisfies the usual asymptotic

requirements and the finiteness at the origin and infinity for the bound state, i.e. we require $R(0) = 0$ and $R(\infty) \to 0$. So, by Eq. (8), we obtain

$$\frac{d^2 R_{nm}(r)}{dr^2} = \left[g''_m(r) + g'^2_m(r) + \frac{f''_n(r) + 2f'(r)g'(r)}{f_n(r)} \right] R_{nm}(r)\,. \tag{11}$$

For case $n = 0$, substituting $f_0(r)$ and $g_m(r)$ in the above equation and comparing the obtained equation with Eq. (5) leads to

$$\left[-p - \frac{\delta}{r^2} + p^2 r^2 + q^2 + \frac{\delta^2}{r^2} + 2pqr - 2p\delta - \frac{2q\delta}{r} \right] + \left[\varepsilon - \gamma^2 r^2 - b_1 r + \frac{c_1}{r} - \frac{m'^{2-1}}{r^2} \right] = 0\,. \tag{12}$$

By equating the corresponding powers of r on Eq. (12), one can obtain

$$\varepsilon + q^2 - p(2\delta + 1) = 0\,, \tag{13a}$$

$$\delta(\delta - 1) - \left(m'^2 - \frac{1}{4} \right) = 0\,, \tag{13b}$$

$$p^2 - \gamma^2 = 0\,, \tag{13c}$$

$$2pq - b_1 = 0\,, \tag{13d}$$

$$2q\delta - c_1 = 0\,. \tag{13e}$$

To have well-behaved solutions at boundaries, namely the origin ($r = 0$) and infinity ($r \to \infty$), we can conveniently obtain, from Eqs. (13b) and (13c), the wave function Ansatz parameters as $\delta = |m'| + \frac{1}{2}$ and $p = \sqrt{(\frac{\mu\omega_c}{2})^2 + 2\mu a} > 0$, respectively. Therefore, from (13d) and (13e), one can obtain restriction on the potential parameters as $b = \frac{c\sqrt{(\frac{\mu\omega_c}{2})^2 + 2\mu a}}{|m'| + \frac{1}{2}}$. Finally, the ground state energy eigenvalues formula is found via Eqs. (13a), (6) and (7) as

$$E_{0m} = -\frac{2\mu c^2}{(2|m'| + 1)^2} + \frac{1}{\mu}\sqrt{\left(\frac{\mu\omega_c}{2}\right)^2 + 2\mu a}(|m'| + 1) + \frac{\omega_c m'}{2}\,. \tag{14}$$

Some numerical results are shown in Figs. 1 and 2. We use parameter set as: $a = 0.04$, $c = 1$ and in a.u. units as $\hbar = 2\mu = 1$.[29–31] In Fig. 1, we plotted the variations of eigenenergy as a function of applied strong magnetic fields for different values of AB flux fieldsξ. It is seen from Fig. 1 that the dependence of E_{01} on ω_c is linear for large applied magnetic fields.[39,41,42] In Fig. 2, we plotted the variations of eigen energy as a function of applied small magnetic field values and it is seen that the dependence of E_{01} on ω_c is nonlinear for small applied magnetic fields.[39,41,42]

It is clear from Figs. 1 and 2 that the energy levels are dependent on magnetic field (B) along the zz-direction and AB flux field (α) created from infinitely long solenoid inserted inside the potential which can only contribute a phase outside the solenoid. Consequently, the presence of AB flux field affects only the magnetic

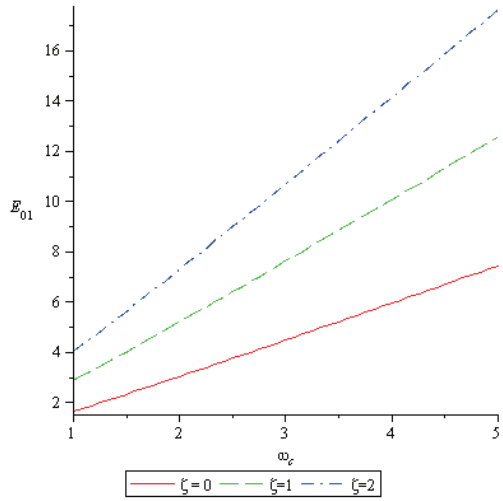

Fig. 1. (Color online) Eigenenergies in a.u. as a function of strong magnetic field values with different AB flux fields.

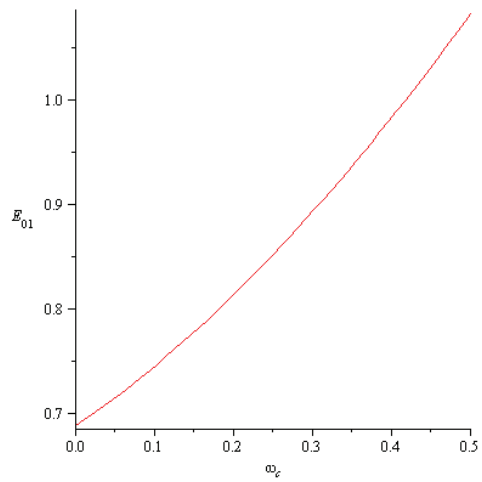

Fig. 2. Eigen energies in a.u. as a function of small magnetic field values.

quantum number and the presence of magnetic field affects only oscillator frequency in the system.

Finally, the normalized eigenfunctions are given as

$$\psi_{0m}(r,\phi) = \frac{N_{0m}}{\sqrt{2\pi}} r^{|m'|} \exp\left[-\left(\frac{1}{2}r^2 + \frac{4c}{2|m'|+1}r\right)p\right] e^{im\phi}\,, \tag{15}$$

where N_{0m} is the normalization factor. It is obvious from above result that $\psi_{0m}(r \to \infty, \phi) \to 0$ and $\psi_{0m}(0,\phi) = 0$.

In the presence of external magnetic and AB flux fields, when $a = \mu\omega^2/2$ and $b = c = 0$, we obtain via Eq. (14) the energy spectrum formula for harmonic oscillator is

$$E_{0m} = \frac{1}{2}\omega_c m' + \frac{1}{2}(|m'|+1)\sqrt{\omega_c^2 + 4\omega^2}\,, \tag{16}$$

which is identical to Eq. (33) of Ref. 40. The wave function becomes

$$\psi_{0m}(r,\phi) = \frac{N_{0m}}{\sqrt{2\pi}} r^{|m'|} \exp\left(-\frac{p}{2}\,r^2\right) e^{im\phi}\,. \tag{17}$$

3. Some Special Cases without External Fields

In this section, we will find the energy levels in absence of external magnetic and AB flux fields and the energy levels in 3D. Coulomb and harmonic oscillator are two special cases to see the accuracy of our results.

First, in the absence of external magnetic fields, i.e. $B = \Phi_{\rm AB} = 0$, the energy equation (14) reduces to

$$E_{0m} = -\frac{2\mu c^2}{(2|m|+1)^2} + \sqrt{\frac{2a}{\mu}}(|m|+1)\,. \tag{18}$$

Second, above energy equation reduces to 3D space if one replaces m by $l + 1/2$ as[37,43,44]

$$E_{0l} = \sqrt{\frac{2a}{\mu}}\left(l + \frac{3}{2}\right) - \frac{\mu c^2}{2\,(l+1)^2} = \sqrt{\frac{2a}{\mu}}\left(l + \frac{3}{2}\right) - \frac{b^2}{4a}\,. \tag{19}$$

It is to be noted that the above analytical result reduces to the result in Ref. 44 when one is replacing $n = 0$ and $N = 3$ in Eq. (29). It can also be reduced into the existing literature results.[45–47] when $n \to 0$ and $\mu = 1$.

The comparison of the eigenvalues E_{0l} (in a.u., $\hbar = 2\mu = 1$) between this work and the analytical results from Refs. 29 and 31 treated by the power series method, the supersymmetric perturbation theory and differential quadratic method, respectively, is listed in Tables 1 and 3. In the light of this, we observe that the results found for different values of the parameters (a; b; c) are in fair agreement with those taken in Refs. 29–31.

Table 1. Bound state energy eigenvalues of the Killingbeck potential in a.u. with $c = 1$ for $l = 0$.

a (Ref. 31)	b	b (Ref. 31)	E_{00}	E_{00} (Ref. 31)	E_{00} (Refs. 29 and 30)
0.01	0.1	0.1	0.05	0.0499	0.05
0.04	0.2	0.2	0.35	0.35	0.35
4	2.0	2.0	5.75	5.749	5.75
100	10	10	29.75	29.75	29.75

Table 2. Bound state energy eigenvalues of the Killingbeck potential in a.u. with $c = 1$ for $l = 1$.

a (Ref. 31)	b	b (Ref. 31)	E_{01}	E_{01} (Ref. 31)	E_{01} (Refs. 29 and 30)
0.01	0.05	0.05	0.4375	0.43749	0.4375
0.04	0.1	0.1	0.9375	0.9377	0.9375
4	1	1	9.9375	9.9376	9.9375
100	5	5	49.9375	49.9375	49.9375

Table 3. Bound state energy eigenvalues of the Killingbeck potential in a.u. with $c = 1$ for $l = 2$.

a (Ref. 31)	b	b (Ref. 31)	E_{02}	E_{02} (Ref. 31)	E_{02} (Refs. 29 and 30)
0.01	0.0333	0.0333	0.67222	0.67277	0.67277
0.04	0.0666	0.0666	1.372222	1.372276	1.372277
4	0.6666	0.6666	13.97222	13.97227	13.97227
100	3.3333	3.333	69.972222	69.972226	69.972227

Third, when potential constant $a = b = 0$, the Killingbeck potential reduces to the Coulomb potential and its energy levels become as[1,2]

$$E_{0l}^{Coul} = -\frac{1}{2}\frac{\mu c^2}{(l+1)^2}. \tag{20}$$

Finally, when potential parameters $b = c = 0$, the Killingbeck potential reduces to the harmonic oscillator potential and its energy levels turns out to be[1,2,40]

$$E_{0l_{\text{H.O.}}} = \omega\left(l + \frac{3}{2}\right), \tag{21}$$

where we put $a = \frac{1}{2}\mu\omega^2$.

4. Conclusions and Remarks

By using the wave function Ansatz method, we investigated the ground states energy levels and wave function for arbitrary m-state in 2D Schrödinger equation with a Killingbeck potential under the influence of strong/weak external uniform magnetic and AB flux fields We obtained the energy levels of Killingbeck potential in the absence of external fields and compared with those ones obtained by other methods. Finally, we found the energy levels of the familiar Coulomb and Harmonic oscillator potentials to show the accuracy of our results.

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References

1. L. I. Schiff, *Quantum Mechanics*, 3rd edn. (McGraw-Hill Book Co., New York, 1955).
2. R. L. Liboff, *Introductory Quantum Mechanics*, 4th edn. (Addison Wesley, San Francisco, CA, 2003).
3. M. M. Nieto, *Am. J. Phys.* **47** (1979) 1067.
4. S. M. Ikhdair and R. Sever, *J. Mol. Struct. (Theochem.)* **806** (2007) 155.
5. S. M. Ikhdair and R. Sever, *J. Mol. Struct. (Theochem.)* **855** (2008) 13.
6. W. C. Qiang and S. H. Dong, *Phys. Lett. A* **363** (2007) 169.
7. S. H. Dong, D. Morales and J. Garcia-Ravelo, *Int. J. Mod. Phys. E* **16** (2007) 189.
8. S. M. Ikhdair and R. Sever, *Int. J. Theor. Phys.* **46** (2007) 1643.
9. S. Ozcelik and M. Simsek, *Phys. Lett. A* **152** (1991) 145.
10. K. J. Oyewumi, *Found. Phys. Lett.* **18** (2005) 75.
11. O. Bayrak, I. Boztosun and H. Ciftci, *Int. J. Quantum Chem.* **107** (2007) 540.
12. M. R. Setare and E. Karimi, *Phys. Scr.* **75** (2007) 90.
13. S. H. Dong, *Phys. Scr.***64** (2001) 273.
14. S. H. Dong, *Int. J. Theor. Phys.* **39** (2000) 1119.
15. C. Berkdemir, A. Berkdemir and J. Han, *Chem. Phys. Lett.* **417** (2006) 326.
16. S. M. Ikhdair and R. Sever, *Cent. Eur. J. Phys.* **5** (2007) 516.
17. M. Hamzavi and A. A. Rajabi, *Int. J. Quantum Chem.* **112** (2012) 1592.
18. J. Killingbeck, *Phys. Lett. A* **65** (1978) 87; J. Killingbeck, *ibid.* **67** (1978) 13.
19. M. K. Bahar and F. Yasuk, *Adv. High Energy Phys.* **2013** (2013) 814985.
20. R. P. Saxena and V. S. Varma, *J. Phys. A: Math. Gen.* **15** (1982) L221.
21. M. Alberg and L. Wilets, *Phys. Lett. A* **286** (2001) 7.
22. R. N. Chaudhuri and M. Mondal, *Phys. Rev. A* **52** (1995) 1850.
23. E. Eichten and K. Gottfried, *Phys. Rev. D* **17** (1978) 3090.
24. E. R. Vrscay, *Phys. Rev. A* **31** (1985) 2054.
25. R. K. Roychoudhury and Y. P. Varshni, *J. Phys. A: Math. Gen.* **21** (1988) 3025.
26. D. Bessis, E. R. Vrscay and C. R. Handy, *J. Phys. A: Math. Gen.* **20** (1987) 419.
27. G. Plante and A. F. Antippa, *J. Math. Phys.* **46** (2005) 062108.
28. Z. Ghalenovi, A. A. Rajabi and M. Hamzavi, *Acta Phys. Pol. B* **42** (2011) 1849.
29. S. C. Chhajlany and A. Letov, *Phys. Rev. A* **44** (1991) 4725.
30. O. Özer and B. Gönül, *Mod. Phys. Lett. A* **18** (2003) 2581.
31. Y. Boumedjane, H. Saidi, S. Hassouni and A. Zerarka, *Appl. Math. Comput.* **194** (2007) 243.
32. M. Aygun, Y. Sahin and I. Boztosun, *Int. J. Mod. Phys. E* **19** (2010) 1349.
33. M. Hamzavi and A. A. Rajabi, *Commun. Theor. Phys.***55** (2011) 35.
34. M. Hamzavi, S. M. Ikhdair and K.-E. Thylwe, *Z. Naturforsch. A* **67** (2012) 567.
35. S. H. Dong, *Phys. Scr.* **65** (2002) 289.
36. S. H. Dong, Z. Q. Ma and G. Sposito, *Found. Phys. Lett.* **12** (1999) 465.
37. S. H. Dong, *Int. J. Theor. Phys.* **40** (2001) 559.
38. W. Greiner, *Relativistic Quantum Mechanic: Wave Equations* (Springer-Verlag, Berlin, 2000).
39. R. Khordad, *Solid State Sci.* **12** (2010) 1253.
40. S. M. Ikhdair and M. Hamzavi, *Chin. Phys. B* **21** (2012) 110302.
41. S. M. Ikhdair, M. Hamzavi and R. Sever, *Phys. B* **407** (2012) 5423.
42. A. Çetin, *Phys. Lett. A* **372** (2008) 3852.
43. M. Hamzavi, H. Hassanabadi and A. A. Rajabi, *Few-Body Syst.* **46** (2009) 183.
44. R. Kumar and F. Chand, *Commun. Theor. Phys.* **59** (2013) 528.
45. R. N. Choudhury and M. Mondal, *Phys. Rev. A* **52** (1995) 1850.

46. S. M. Ikhdair and R. Sever, *J. Mol. Struct.* (*Theochem.*) **806** (2007) 155; S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. C* **18** (2007) 1571; S. M. Ikhdair and R. Sever, *ibid.* **19** (2008) 221; S. M. Ikhdair and R. Sever, *ibid.* **6** (2008) 685; S. M. Ikhdair and R. Sever, *Ann. Phys.* **17** (2008) 897.
47. R. Kumar and F. Chand, *Chin. Phys. Lett.* **29** (2012) 060306.