Z. Phys. C – Particles and Fields 56, 155–160 (1992)

Zeitschrift für Physik C Particles and Fields

Heavy-quark bound states in potentials with the Bethe–Salpeter equation

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Received 21 April 1992

Abstract. The spinless Bethe–Salpeter equation is solved for three attractive static quark–antiquark potentials of the form $V(r) = -ar^{-\beta} + br^{\beta} + c$, $0 < \beta \leq 1$, and the effective non-Coulombic power-law potential of the form $V(r) = ar^{0.1} + c$ to obtain the spin-averaged energy levels in bottomonium $\Upsilon(b\bar{b})$ and charmonium $\psi(c\bar{c})$ families. The shifted $1/N$ expansion technique is used. Calculations of the energy eigenvalues are carried out up to third order and parameters of each potential are adjusted to obtain the best agreement with the experimental spin-averaged data (SAD). Flavor-dependent and flavor-independent cases are considered in this work.

1 Introduction

The Bethe–Salpeter (BS) equation [1–5] is appealing because it is a fully covariant equation describing the interaction of a pair of elementary particles and has a formal connection with quantum field theory and perturbation theory. The BS equation is a more realistic starting point for the relativistic case than the Schrödinger equation with $(v/c)^2$ correction for the light as well as the heavy-quark systems. While the BS equation, being an invariant two-body equation may seem as an appropriate formalism for the study of the relativistic quark–antiquark confined system, its use in such a context has long been known to present several difficulties in its solution.

A standard approximation [3] to this equation has been obtained by replacing the interaction kernel by an instantaneous local potential and by neglecting spin and coupling of the "large–large" and "small–small" components of the wave function. This equation is suitable for describing the spin-averaged spectrum of two bound, interacting quark–antiquark of masses m_1 and m_2 and $M(Q\bar{Q})$.

With this approximation, Nickisch et al. [3] presented and tested two methods for the numerical solution of BS equation. The BS equation has also been solved numerically and analytically for its bound-state energies using different techniques. The first numerical technique [6] is to Fourier-transform the equation and work it out in momentum space. The second numerical technique [2, 7] is to expand the wave function in some complete set of basis states for which the term in BS equation is simple. B. Durand and L. Durand [8] constructed an analytic solution to the spinless S-wave Salpeter equation for two fermions interacting via a Coulombic potential by transforming the momentum-space form of the equation into a mapping or a boundary value problem for analytic wave functions. Cea et al. [9] developed a WKB approach to the solution of this kind of problems and extended [7] their analysis to a larger class which includes singular potentials. Jacobs et al. [10] followed [4, 5] and solved BS equation numerically in momentum space. The aim of this paper is to apply our approach for some well-known static and power-law quarkonium potentials [11–17] to predict the spin-averaged energy levels and bound-state masses.

In Sect. 2 we briefly discuss the wave equation used and its analytic solution. In Sect. 3 we describe the different static potentials used in our calculations and present our numerical results for the $c\bar{c}$ and $b\bar{b}$ bound-state masses. Sect. 4 presents a discussion of the results.

2 Analytic solution of the Bethe–Salpeter equation

We consider the analytic solution of general coordinate-space relativistic spinless Bethe–Salpeter (BS) equation for a two-body system

$$[(-\nabla^2 + m_1^2)^{1/2} + (-\nabla^2 + m_2^2)^{1/2} + V(r) - M(Q\bar{Q})]\psi(\mathbf{r}) = 0, \tag{1}$$

whose radial part in the N-dimensional space, for the two-body system of equal masses, expanded in powers of v^2/c^2 up to two terms, is [3, 4]

$$\left\{-\frac{\nabla_N^2}{m} + \frac{\nabla_N^4}{4m^3} + V(r)\right\} R(r) = E_{nl} R(r), \tag{2}$$

where the operator ∇_N^2 is written as

$$\nabla_N^2 = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{L_{N-1}^2}{r^2}. \tag{3}$$

Making the substitution

$$R(r) = r^{-(N-1)/2} U(r) \tag{4}$$

and factorizing the resulting fourth-order equation into two second-order equations, one gets a second-order Schrödinger-like equation to order v^2/c^2 perturbation term:

$$\left\{ -\frac{1}{2m}\frac{d^2}{dr^2} + \frac{[\bar{k}-(1-a)][\bar{k}-(3-a)]}{8mr^2} + m \mp m\left[1 + \frac{E_{nl}-V(r)}{m} + \frac{(E_{nl}-V(r))^2}{2m^2} + \cdots\right]\right\} \hat{U}(r) = 0, \tag{5}$$

where $\hat{U}(r)$ is the new reduced wave function given by

$$\hat{U}(r) = \frac{1}{2m}\left\{ -\frac{1}{2m}\frac{d^2}{dr^2} + \frac{[\bar{k}-(1-a)][\bar{k}-(3-a)]}{8mr^2} + m \pm m\left[1 + \frac{E_{nl}-V(r)}{m} + \frac{(E_{nl}-V(r))^2}{2m^2} + \cdots\right]\right\} U(r). \tag{6}$$

The $[E_{nl}-V(r)]^2$ perturbation term in (5) is significant only where it is small, $[E_{nl}-V(r)]/2m \ll 1$. Employing the expansions presented in [18] for the potential $V(r)$ as well as for the total energy E_{nl}, one solves (5) and gets

$$\varepsilon_{nr} = \frac{r_0^2}{Q}\left[\bar{k}\left(E_0 + \frac{E_0^2}{2m}\right) + \left(E_1 + \frac{E_0 E_1}{m}\right) + \left(E_2 + \frac{E_0 E_2}{m} + \frac{E_1^2}{2m}\right)\frac{1}{\bar{k}^2} + \left(E_3 + \frac{E_0 E_3}{m} + \frac{E_1 E_2}{m}\right)\frac{1}{\bar{k}^2} + \cdots\right], \tag{7}$$

where n_r is the radial quantum number. Comparing (5) with its counterpart Schrödinger-like equation for the one-dimensional anharmonic oscillator problem, discussed before by Imbo et al. [19], and making use of (7), we get

$$\left(\frac{r_0^2}{Q}\right)\left[E_0 + \frac{E_0^2}{2m}\right] = \left[\frac{1}{8m} + \frac{r_0^2 V(r_0)}{Q} - \frac{r_0^2 V(r_0)^2}{2mQ} + \frac{r_0^2 E_0 V(r_0)}{mQ}\right], \tag{8}$$

$$\left(\frac{r_0^2}{Q}\right)\left[E_1 + \frac{E_0 E_1}{m}\right] = \left[(1+2n_r)\frac{\omega}{2} - \frac{(2-a)}{4m}\right], \tag{9}$$

$$\left(\frac{r_0^2}{Q}\right)\left[E_2 + \frac{E_0 E_2}{m} + \frac{E_1^2}{2m}\right] = \left[\frac{r_0^2 E_2 V(r_0)}{mQ} + \beta^{(1)}\right], \tag{10}$$

$$\left(\frac{r_0^2}{Q}\right)\left[E_3 + \frac{E_0 E_3}{m} + \frac{E_1 E_2}{m}\right] = \left[\frac{r_0^2 E_3 V(r_0)}{mQ} + \beta^{(2)}\right], \tag{11}$$

where $\beta^{(1)}$ and $\beta^{(2)}$ are defined in Appendix A. Accordingly, (8) yields

$$E_0 = V(r_0) - m + m(1 + Q/4m^2 r_0^2)^{1/2}, \tag{12}$$

where r_0, the minimum of the effective energy E_0 [19, 20], is determined by the following conditions:

$$\frac{dE_0}{dr_0} = 0, \qquad \frac{d^2 E_0}{dr_0^2} > 0. \tag{13}$$

The next contribution to the energy is chosen to be zero [19], i.e., $E_1 = 0$, to fix the proper shift so as to store the contribution of higher-order terms in the leading order of the expansion term E_0, for which

$$a = 2 - 2m(1+2n_r)\omega, \tag{14}$$

where ω is defined by

$$\omega = \frac{1}{2m}[3 + r_0 V''(r_0)/V'(r_0) - 4r_0^4 V'(r_0)^2/Q]^{1/2}, \tag{15}$$

and

$$Q = [r_0^2 V'(r_0)]^2(2+2\eta), \tag{16}$$

with

$$\eta = [1 + [2m/r_0 V'(r_0)]^2]^{1/2}. \tag{17}$$

Once r_0 is determined from

$$1 + 2l + 2m(2n_r+1)\omega = r_0^2 V'(r_0)(2+2\eta)^{1/2}, \tag{18}$$

it becomes easy and straightforward to obtain the total binding energy of the $Q\bar{Q}$ system by

$$E_{nl} = E_0 + \frac{\beta^{(1)}}{r_0^2\left(1 + \frac{E_0 - V(r_0)}{m}\right)} + \frac{\beta^{(2)}}{r_0^2\left(1 + \frac{E_0 - V(r_0)}{m}\right)\bar{k}} + O\left(\frac{1}{\bar{k}^2}\right). \tag{19}$$

The quantities $\beta^{(1)}$ and $\beta^{(2)}$ appearing in the corrections to the leading order of the energy expansion are defined as in [21] and are listed in Appendix A. The calculated analytic expressions for the δ's and ε's in their detailed expressions, appearing in $\beta^{(1)}$ and $\beta^{(2)}$, are also listed in Appendix A.

The quarkonium energy levels $M(Q\bar{Q})$ are then given by

$$M(Q\bar{Q}) = 2(E_{nl} + m_Q), \tag{20}$$

where m_Q stands for either m_c or m_b.

We solve (5) numerically by making use of (12) and (19) repeatedly with each potential and then calculate (20) considering the combined parameter set of $\Upsilon(b\bar{b})$ and $\psi(c\bar{c})$ spectra as a flavor-independent case and separated parameter set of $\Upsilon(b\bar{b})$ and $\psi(c\bar{c})$ spectra as a flavor-dependent case.

3 Static quarkonium potentials

We investigate four different potentials, one of which is a QCD-motivated Cornell potential [11], and the other three of them are strictly phenomenological potentials such as the Song–Lin [12], Turin [13, 14] and Martin [15] potentials.

a) Cornell potential. The Cornell potential [11] is

$$V_C = -a/r + br + c, \tag{21}$$

where $a = 4\alpha_s/3$ is supposed to represent the short-range gluon exchange and b is the confinement string tension. The main drawback of this potential is that the $c\bar{c}$ and $b\bar{b}$ states lie in an intermediate region of quark separation where none of the limiting forms of (21) should be valid [10].

b) Song–Lin potential. This phenomenological potential proposed by Song and Lin and has the form

$$V_{SL} = -ar^{-1/2} + br^{1/2} + c, \tag{22}$$

where a, b and c are adjustable parameters. The characteristic feature of this potential can be traced from [12].

c) Turin potential. Lichtenberg and coworkers [13] suggested a new potential which is an intermediate between the Cornell and Song–Lin potentials and called it Turin potential [14]. This potential has the form

$$V_T = -ar^{-3/4} + br^{3/4} + c. \tag{23}$$

d) Martin potential. The last phenomenological potential we study is the power-law potential [16]

$$V(r) = ar^{\nu} + c. \tag{24}$$

The logarithmic potential may also be regarded as the limit of either a nonsingular ($\nu > 0$) or singular ($\nu < 0$) potential and has the form [16]

$$V(r) = c \ln(r/r_0), \tag{25}$$

where $\nu \approx 0$. Here we investigate a simple case of (24) with $\nu = 0.1$, written as

$$V_M = ar^{0.1} + c, \tag{26}$$

considered previously by Martin [15].

The first three potentials are considered as special cases of a class of potentials

$$V = -ar^{-\beta} + br^{\beta} + c, \quad 0 < \beta \leq 1, \tag{27}$$

previously proposed in [17] for quarkonium. These static quarkonium potentials are monotone nondecreasing and concave functions of r and satisfy the condition [14]

$$\frac{dV}{dr} > 0, \quad \frac{d^2V}{dr^2} \leq 0. \tag{28}$$

We compute the bound-state masses of $\Upsilon(b\bar{b})$ and $\psi(c\bar{c})$ with each of the four potentials using the parameters of [13, 14] for the sake of comparison. We also consider the flavor dependence and flavor independence of these potentials with respect to the separated and combined calculated spin-averaged data (SAD) of [14].

4 Results and conclusions

The Bethe–Salpeter two-body equation has been solved with four different potentials using the powerful shifted large-N expansion technique for the spin-averaged quarkonium energy levels. The calculations are carried out up to third order in the energy series. The results are presented in Tables 1–4. The least-square fit (chi-square test) χ^2 values [22, 23] are calculated and compared, for each potential, for the flavor-dependent and flavor-independent cases.

In this study when the $\psi(c\bar{c})$ and $\Upsilon(b\bar{b})$ combined and separated data are considered with flavor-independent

Table 1. Calculated spin-averaged data (SAD) of bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ energy levels in MeV, together with the least-square fit χ^2 [a] for the case of relativistic Bethe–Salpeter equation with flavor-independent potentials

		SAD [b]	Cornell [c]	Song–Lin [c]	Turin [c]
Combined					
$b\bar{b}$	1S	9446±3	9447	9497	9472
	2S	10017±3	10015	10037	10023
	3S	10350±3	10352	10356	10351
	4S	10576±7	10621	10592	10602
	5S	10861±11	10854	10785	10812
	6S	11016±11	11064	10949	10998
	1P	9900±2	9910	9906	9905
	2P	10261±2	10250	10259	10253
χ_b^2			117	437	137
$c\bar{c}$	1S	3068±1	3069	3073	3068
	2S	3663±2	3638	3633	3633
	3S	4028±12	4039	3979	4004
	4S	4405±8	4373	4242	4299
	1P	3525±2	3483	3491	3488
	1D	3788±7	3795	3764	3762
	2D	4176±24	4115	4064	4087
χ_c^2			854	988	774
$\chi^2 = \chi_c^2 + \chi_b^2$			971	1425	911

[a] Here we cite [22]; [b] Here we cite [14]; [c] present work

Table 2. Calculated spin-averaged data (SAD) of bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ energy levels in MeV, together with the least-square fit χ^2 for the case of relativistic Bethe–Salpeter equation with flavor-dependent potentials

		SAD [a]	Cornell [b]	Song–Lin [b]	Turin [b]
$b\bar{b}$	1S	9446±3	9455	9478	9467
	2S	10017±3	10013	10026	10017
	3S	10350±3	10351	10346	10345
	4S	10576±7	10620	10582	10596
	5S	10861±11	10853	10773	10807
	6S	11016±11	11064	10936	10992
	1P	9900±2	9907	9895	9900
	2P	10261±2	10247	10249	10247
χ_b^2			131	284	138
$c\bar{c}$	1S	3068±1	3072	3078	3077
	2S	3663±2	3639	3643	3638
	3S	4028±12	4025	4005	4009
	4S	4405±8	4342	4283	4304
	1P	3525±2	3498	3484	3491
	1D	3788±7	3762	3764	3764
	2D	4176±24	4102	4085	4090
χ_c^2			427	883	713
$\chi^2 = \chi_c^2 + \chi_b^2$			558	1167	851

[a] Here we cite [14]; [b] present work

Table 3. Calculated spin-averaged data (SAD) of bottomonium $b\bar{b}$ energy levels in MeV, together with the least-square fit χ^2 [a] for the case of relativistic Bethe–Salpeter equation with the Martin potential

State	SAD[b]	Ref. [13]	Martin[c]
$1S$	9446 ± 3	9460	9451
$2S$	10017 ± 3	10038	10023
$3S$	10350 ± 3	10374	10355
$4S$	10576 ± 7	10615	10592
$5S$	10861 ± 11	10804	10779
$6S$	11016 ± 11	10960	10933
$1P$	9900 ± 2	9878	9869
$2P$	10261 ± 2	10263	10250
$7S$	—	11094	11065
$3P$	—	10530	10512
$1D$	—	10152	10143
$2D$	—	10444	10431
χ_b^2		341	398

[a] Here we cite [22]; [b] here we cite [14]; [c] present work

Table 4. Fit to spin-averaged levels using the Cornell potential of the form $V(r) = -\kappa/r + \beta r$, together with the least-square fit χ^2. Level energies are in MeV. The potential parameters and quark masses were varied to account best for the spin-averaged data (SAD)

State	SAD[a]	Ref. [10]	Present work
$c\bar{c}$			
M_{1S}	3068 ± 1	3069	3067
M_{1P}	3525 ± 2	3506	3504
M_{2S}	3663 ± 2	3667	3682
χ_c^2		94	201
$b\bar{b}$			
M_{1S}	9446 ± 3	9445	9443
M_{1P}	9900 ± 2	9903	9902
M_{2S}	10017 ± 3	10000	10020
M_{2P}	10261 ± 2	10263	10262
M_{3S}	10350 ± 3	10350	10377
χ_b^2		34	84
Fit parameters			
κ	0.446		
β (GeV)^2	0.200		
m_c (GeV)	1.327		
m_b (GeV)	4.737		

[a] Here we cite [14]

and flavor-dependent potentials, we obtain some insight into the difficulty of exactly accounting for the SAD. To determine whether a flavor-dependent potential is significantly better than a flavor-independent potential, we compared the χ^2 value of the flavor-independent one with that of the flavor-independent one. Based on our results, for the Cornell potential we see from Tables 1 and 2 that χ^2 is much lower for the flavor-dependent case. We conclude that in this case the flavor-dependent Cornell potential accounts for the SAD fairly well than compared to the flavor-independent one. Moreover, the differences in χ^2 for the Song–Lin and the Turin potentials are not so large; nevertheless, the flavor-dependent version of each potential considered is significantly better than the flavor-independent one. For the Turin potential the value of χ^2 for the fits to the combined data with flavor-independent potential is about the same as the flavor-dependent one. In this case, the potential can be taken to be independent of the flavor, without deterioration in the fits to the data.

On the other hand, even the flavor-dependent versions of the three potentials do not exactly account for the SAD. For a satisfactory agreement between theory and experiment, for a flavor-dependent potential we expect χ^2 to be small, but we see from Table 2 that the best flavor-dependent potential (the Cornell potential) gives the lowest χ^2. The best flavor-independent potential (the Turin and the Cornell potentials) provide considerably poorer fits as seen in Table 1. Indeed, numerical experimentation with these potentials, characterized by three parameters, can provide a satisfactory fit to the data [23].

We now call our attention to the following additional features of the fits extracted from Tables 1 and 2:

1) The phenomenological Turin potential with $\beta = 3/4$ and the QCD-motivated Cornell potential are most suitable for bottomonium in flavor-dependent and flavor-independent cases, with a slightly better accuracy for the Cornell potential in the flavor-independent case.
2) The QCD-motivated Cornell potential is by far the best for charmonium in flavor-dependent case.
3) The flavor-independent Turin potential is essentially as good as the flavor-dependent Turin potential, while the Song–Lin flavor-dependent potential is slightly better than the flavor-independent one and provides a considerably poorer fit to the data.

We have also the following comments and main features:

1) The parameters of the potentials and the χ^2 values are significantly changed for bottomonium as well as for charmonium, a fact that shows the effects of relativistic effects are not negligible even in bottomonium. Since the charmonium states are expected to have significant relativistic corrections, a relativistic treatment may also be expected to restore flavor independence.
2) Surprisingly, for each potential the fits to the bottomonium SAD are much better than those to charmonium, since bottomonium has a larger mass, for which the perturbation terms in (5) become more significant where they are small, a case already satisfied for heavier quarkonium systems.

In this paper, when we looked for best fits to the data for flavor-dependent and flavor-independent potentials, we found that none of the potentials considered in this work gives an adequate account of either the charmonium or the bottomonium SAD. It is believed that this failure to fit the data quantitatively is not just a defect in the potential used, but is a characteristic phenomenon of all simple concave downward static potentials, characterized by just a few parameters, as stated by Lichtenberg and his coworkers. Moreover, the data include some theoretical input, and have included sufficiently large theoretical contributions to the errors in the energies of the spin-averaged levels. Measurements of the missing energy levels in bottomonium and charmonium would be extremely valuable to enable one to sharpen the analysis. The prescription and

the wave equation used are responsible for the deterioration of the fit parameters. The calculations and parameters are model-dependent.

In Table 3 we presented the calculated energies of the SAD for the Martin potential, including predictions of the energies of the $7S$, $3P$, $1D$ and $2D$ levels, which have not yet been observed. The Martin potential also provides quite good predictions of the SAD as well as the other static quarkonium potentials. The value of χ^2 is found to be the same as that in [13] and the predictions for the lower states of bottomonium are considerably better than those of [13] since the $\Upsilon(nS)$, where $n \geq 4$, lie above the threshold and predominantly decay into mesons containing beauty flavor.

The analysis of the Cornell potential was repeated for another two-parameter Cornell potential [10], as shown in Table 4. Our results with such a form of the potential were stringent and account for the SAD fairly well and were within the range of the errors. By comparing Tables 1, 2 and 4 in the case of the Cornell potential, we remark that the parameters of the potential depend on the wave equation and the form of the relativistic kinematics used. Moreover, our results are in very good agreement with the results presented in [10] since both the works considered the same BS equation, whereas [14] has considered a Schrödinger equation with relativistic kinematics. Although our treatment is completely different from that presented in [14], it appears that the calculated SAD are in good agreement.

The essential character of this investigation is that none of these potentials provides an adequate account of either the charmonium ($c\bar{c}$) or the bottomonium ($b\bar{b}$) spin-averaged data. Only rough estimations of the SAD have been carried out since the energies of some of the singlet states of these families have not been measured. Such a theoretical estimation of the unknown levels introduces an uncertainty into the SAD which is appreciably larger than the experimental errors in the measured levels. Therefore, we corroborate the comments of Lichtenberg et al. [14] that no static potential can reproduce the observed spin-averaged energy levels in quarkonium within the experimental errors.

The overall results have very high accuracy and show that the shifted $1/N$ expansion technique works well for all spherically symmetric quarkonium potentials, encouraging the use of such an approach for more complicated potentials. In general, the shifted method, though an efficient quantitative method, is able to produce only reasonable approximations for the spin-averaged bound states of the quarkonium. We may point out that the higher corrected terms in powers of v^2/c^2 and the Breit–Fermi terms [24], which could account better for the SAD, have not been considered. A natural extension of the present work is an incorporation of these relativistic corrections and spin dependence. The relativistic and spin-dependent parts of the $Q\bar{Q}$ interaction can be obtained in terms of the scalar and vector potentials by means of a reduction of the present equation [24]. For the short-range potential, which is not very sensitive to the spin-averaged states or leptonic widths, one expects that the potentials would be determined better than our SAD case since the spin-dependent parts depend on derivatives of the potentials.

Appendix A: Analytic expressions of $\beta^{(1)}$, $\beta^{(2)}$, ε_I and δ_J in case of the spinless Bethe–Salpeter equation

$$\beta^{(1)} = \frac{(1-a)(3-a)}{8m} + [(1+2n_r)\bar{\varepsilon}_2 + 3(1+2n_r+2n_r^2)\bar{\varepsilon}_4] - \omega^{-1}[\bar{\varepsilon}_1^2 + 6(1+2n_r)\bar{\varepsilon}_1\bar{\varepsilon}_3 + (11+30n_r+30n_r^2)\bar{\varepsilon}_3^2], \tag{A.1}$$

$$\begin{aligned}\beta^{(2)} = {} & (1+2n_r)\delta_2 + 3(1+2n_r+2n_r^2)\bar{\delta}_4 \\ & + 5(3+8n_r+6n_r^2+4n_r^3)\bar{\delta}_6 - \omega^{-1}[(1+2n_r)\bar{\varepsilon}_2^2 \\ & + 12(1+2n_r+2n_r^2)\bar{\varepsilon}_2\bar{\varepsilon}_4 + 2\bar{\varepsilon}_1\bar{\delta}_1 \\ & + 2(21+59n_r+51n_r^2+34n_r^3)\bar{\varepsilon}_4^2 + 6(1+2n_r)\bar{\varepsilon}_1\bar{\delta}_3 \\ & + 30(1+2n_r+2n_r^2)\bar{\varepsilon}_1\bar{\delta}_5 + 6(1+2n_r)\bar{\varepsilon}_3\bar{\delta}_1 \\ & + 2(11+30n_r+30n_r^2)\bar{\varepsilon}_3\bar{\delta}_3 \\ & + 10(13+40n_r+42n_r^2+28n_r^3)\bar{\varepsilon}_3\bar{\delta}_5] \\ & + \omega^{-2}[4\bar{\varepsilon}_1^2\bar{\varepsilon}_2 + 36(1+2n_r)\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3 \\ & + 8(11+30n_r+30n_r^2)\bar{\varepsilon}_2\bar{\varepsilon}_3^2 \\ & + 24(1+2n_r)\bar{\varepsilon}_1^2\bar{\varepsilon}_4 + 8(31+78n_r+78n_r^2)\bar{\varepsilon}_1\bar{\varepsilon}_3\bar{\varepsilon}_4 \\ & + 12(57+189n_r+225n_r^2+150n_r^3)\bar{\varepsilon}_3^2\bar{\varepsilon}_4] \\ & - \omega^{-3}[8\bar{\varepsilon}_1^3\bar{\varepsilon}_3 + 108(1+2n_r)\bar{\varepsilon}_1^2\bar{\varepsilon}_3^2 \\ & + 48(11+30n_r+30n_r^2)\bar{\varepsilon}_1\bar{\varepsilon}_3^3 \\ & + 30(31+109n_r+141n_r^2+94n_r^3)\bar{\varepsilon}_3^4],\end{aligned} \tag{A.2}$$

where

$$\bar{\varepsilon}_i = \frac{\varepsilon_i}{(2m\omega)^{i/2}}, \quad i=1, 2, 3, 4,$$

$$\bar{\delta}_j = \frac{\delta_i}{(2m\omega)^{j/2}}, \quad j=1, 2, 3, 4, 5, 6, \tag{A.3}$$

$$\varepsilon_1 = \frac{(2-a)}{2m}, \quad \varepsilon_2 = -\frac{3}{4m}(2-a), \tag{A.4}$$

$$\varepsilon_3 = -\frac{1}{2m} + \frac{r_0^5}{6Q}\left[V'''(r_0) - \frac{V(r_0)V'''(r_0)}{m} - \frac{3V'(r_0)V''(r_0)}{m} + \frac{V'''(r_0)E_0}{m}\right], \tag{A.5}$$

$$\varepsilon_4 = \frac{5}{8m} + \frac{r_0^6}{24Q}\left[V''''(r_0) - \frac{V(r_0)V''''(r_0)}{m} - \frac{4V'(r_0)V'''(r_0)}{m} - \frac{3V''(r_0)V''(r_0)}{m} + \frac{V''''(r_0)E_0}{m}\right], \tag{A.6}$$

$$\delta_1 = -\frac{(1-a)(3-a)}{4m} + \frac{r_0^3 E_2 V'(r_0)}{mQ}, \tag{A.7}$$

$$\delta_2 = \frac{3(1-a)(3-a)}{8m} + \frac{r_0^4 E_2 V''(r_0)}{2mQ}, \tag{A.8}$$

$$\delta_3 = \frac{(2-a)}{m}, \quad \delta_4 = -\frac{5(2-a)}{4m}, \tag{A.9}$$

$$\delta_5 = \frac{3}{4m} + \frac{r_0^7}{120Q}\left[V'''''(r_0) - \frac{V(r_0)V'''''(r_0)}{m} - \frac{5V'(r_0)V''''(r_0)}{m}\right.$$

$$-\frac{10V''(r_0)V'''(r_0)}{m}+\frac{V''''(r_0)E_0}{m}\Bigg], \tag{A.10}$$

$$\delta_6=\frac{7}{8m}+\frac{r_0^8}{720Q}\Bigg[V''''(r_0)-\frac{V(r_0)V''''''(r_0)}{m}-\frac{6V'(r_0)V''''(r_0)}{m}$$

$$-\frac{10V'''(r_0)V'''(r_0)}{m}-\frac{15V''(r_0)V''''(r_0)}{m}+\frac{V''''''(r_0)E_0}{m}\Bigg]. \tag{A.11}$$

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