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EXACT SOLUTIONS OF THE MODIFIED KRATZER POTENTIAL PLUS RING-SHAPED POTENTIAL IN THE D-DIMENSIONAL SCHRÖDINGER EQUATION BY THE NIKIFOROV–UVAROV METHOD

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We present analytically the exact energy bound-states solutions of the Schrödinger equation in D dimensions for a recently proposed modified Kratzer plus ring-shaped potential by means of the Nikiforov–Uvarov method. We obtain an explicit solution of the wave functions in terms of hyper-geometric functions (Laguerre polynomials). The results obtained in this work are more general and true for any dimension which can be reduced to the well-known three-dimensional forms given by other works.

Keywords: Energy eigenvalues and eigenfunctions; modified Kratzer potential; ring-shaped potential; non-central potentials; Nikiforov and Uvarov method.

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1. Introduction

The important task of quantum mechanics is to find the exact bound-states solution of the Schrödinger equation for certain potentials of physical interest. Generally speaking, there are a few main traditional methods to study the exact solutions of quantum systems like the Coulomb, the harmonic oscillator,[1,2] the pseudoharmonic[3,4] and the Kratzer[4,5] potentials. Additionally, in order to obtain the bound-states solutions of central potentials, one has to resort to numerical techniques or approximation schemes. For many of the quantum mechanical systems, most popular approximation methods such as shifted $1/N$ expansion,[6] perturbation theory,[7] path integral solution,[8] algebraic methods with the SUSY quantum mechanics method and the idea of shape invariance, further closely with the factorization method,[9] exact quantization rule,[10,11] and the Nikiforov and Uvarov (NU) method.[12–24] Some of these methods have drawbacks in applications. Although some other methods give simple relations for the eigenvalues, however, they lead to very complicated relations for the eigenfunctions.

The study of exact solutions of the Schrödinger equation for a class of non-central potentials with a vector potential and a non-central scalar potential is of considerable interest in quantum chemistry.[25–34] In recent years, numerous studies[35–39] have been made in analyzing the bound states of an electron in a Coulomb field with simultaneous presence of Aharanov–Bohm (AB)[40] field, and/or a magnetic Dirac monopole,[41] and Aharanov–Bohm plus oscillator (ABO) systems. In most of these studies, the eigenvalues and eigenfunctions are obtained by means of separation of variables in spherical or other orthogonal curvilinear coordinate systems. The path integral for particles moving in non-central potentials is evaluated to derive the energy spectrum of this system analytically.[42] In addition, the idea of SUSY and shape invariance is also used to obtain exact solutions of such non-central but separable potentials.[43] Very recently, the NU method has been used to give a clear recipe of how to obtain explicit exact bound-states solutions for the energy eigenvalues and their corresponding wave functions in terms of orthogonal polynomials for a class of non-central potentials.[44]

Recently, Chen and Dong[45] found a new ring-shaped potential and obtained the exact solution of the Schrödinger equation for the Coulomb potential plus this new ring-shaped potential, which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene. Very recently, Cheng and Dai,[46] proposed a new potential consisting of the modified Kratzer's potential[47] plus the new proposed ring-shaped potential in Ref. 45. They have presented the energy eigenvalues for this proposed exactly-solvable non-central potential in three-dimensional (i.e. $D = 3$) Schrödinger equation through the NU method. The two quantum systems solved by Refs. 45 and 46 are closely relevant to each other as they deal with a Coulombic field interaction except for a slight change in the angular momentum barrier acts as a repulsive core which is for any arbitrary angular momentum ℓ prevents collapse of the system in any dimensional space due to the slight perturbation to the original angular momentum barrier.

The Nikiforov–Uvarov (NU) method,[12] which received much interest, has been introduced for solving Schrödinger,[13–21] Klein–Gordon,[21,22] Dirac[24] and Salpeter[24] equations. We will follow a parallel solution to Ref. 46 and give complete exact bound-states solutions and normalized wave functions of the D-dimensional Schrödinger equation with modified Kratzer plus ring-shaped potential, a Coulombic-like potential with an additional centrifugal potential barrier, for any arbitrary ℓ'-states using the Nikiforov–Uvarov method. Our general solution reduces to the standard three dimensions given by Ref. 46 in the limiting case of $D = 3$.

This work is organized as follows: in Sec. 2, we shall briefly introduce the basic concepts of the NU method. Section 3 is mainly devoted to the exact solution of the Schrödinger equation in D dimensions for this quantum system by means of the NU method. Finally, the relevant results are discussed in Sec. 4.

2. Basic Concepts of the Method

The NU method is based on reducing the second-order differential equation to a generalized equation of hyper-geometric type.[12] In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. Using a wave function, $\psi_n(s)$, of the simple ansatz:

$$\psi_n(s) = \phi_n(s) y_n(s)\,, \tag{2}$$

reduces Eq. (1) into an equation of a hyper-geometric type

$$\sigma(s) y_n''(s) + \tau(s) y_n'(s) + \lambda y_n(s) = 0\,, \tag{3}$$

where

$$\sigma(s) = \pi(s)\frac{\phi(s)}{\phi'(s)}\,, \tag{4}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s)\,, \qquad \tau'(s) < 0\,, \tag{5}$$

and λ is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s)\,, \qquad n = 0, 1, 2, \ldots\,. \tag{6}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$, which is a polynomial of degree n. Furthermore, the other part $y_n(s)$ of the wave function (2) is the hyper-geometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{7}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition[12]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s)\,, \qquad w(s) = \sigma(s)\rho(s)\,. \tag{8}$$

The function π and the parameter λ are defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}\,, \tag{9}$$

$$\lambda = k + \pi'(s)\,. \tag{10}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (9) can be arranged to be the square of a polynomial of first degree.[12] This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are substituted in Eq. (9). In addition, by comparing Eqs. (6) and (10), we obtain the energy eigenvalues.

3. Exact Solutions of the Quantum System with the NU Method

3.1. *Separating variables of the Schrödinger equation*

The modified Kratzer plus ring-shaped potential in spherical coordinates is defined as[46]

$$V(r,\theta) = D_e\left(\frac{r - r_e}{r}\right)^2 + \beta\frac{\cos^2\theta}{r^2\sin^2\theta}\,, \tag{11}$$

where β is positive real constant. The potential in Eq. (11) introduced by Cheng–Dai[46] reduces to the modified Kratzer potential in the limiting case of $\beta = 0$.[47] In fact the energy spectrum for this potential can be obtained directly by considering it as special case of the general non-central separable potentials.[44]

Our aim is to derive analytically the energy spectrum for a moving particle in the presence of a potential (11) in a very simple way.

We begin by considering the Schrödinger equation in arbitrary dimensions D for our proposed potential:[6,48]

$$-\frac{\hbar^2}{2\mu}\nabla_D^2\psi_{\ell_1\cdots\ell_{D-2}}^{(\ell_{D-1}=\ell)}(\mathbf{x}) = [E - V(r)]\psi_{\ell_1\cdots-\ell_{D-2}}^{(\ell_{D-1}=\ell)}(\mathbf{x})\,, \tag{12}$$

where μ and E denote the reduced mass and energy of two interacting particles, respectively. $\mathbf{x}$ is a D-dimensional position vector with the hyper-spherical Cartesian components $x_1, x_2, \ldots, x_D$ given as follows:[a,48–54]

$$x_1 = r\cos\theta_1\sin\theta_2\cdots\sin\theta_{D-1}\,,$$

$$x_2 = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-1}\,,$$

$$x_3 = r\cos\theta_2\sin\theta_3\cdots\sin\theta_{D-1}\,,$$

[a]It is worth noting that such a definition was introduced by Erdélyi early in the 1950s (cf. Ref. 53, Chapter 11, pp. 232–235) even though the notation used by him is quite different from that by Louck and Chatterjee.

$$\vdots$$

$$x_j = r\cos\theta_{j-1}\sin\theta_j \cdots \sin\theta_{D-1}\,, \qquad 3 \le j \le D-1\,,$$

$$\vdots$$

$$x_{D-1} = r\cos\theta_{D-2}\sin\theta_{D-1}\,,$$

$$x_D = r\cos\theta_{D-1}\,, \qquad \sum_{j=1}^{D} x_j^2 = r^2\,, \tag{13}$$

for $D = 2, 3, \ldots$. We have $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ for $D = 2$, and $x_1 = r\cos\varphi\sin\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\theta$ for $D = 3$. The Laplace operator ∇_D^2 is defined by

$$\nabla_D^2 = \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2}\,. \tag{14}$$

The volume element of the configuration space is given by

$$\prod_{j=1}^{D} dx_j = r^{D-1} dr d\Omega\,, \qquad d\Omega = \prod_{j=1}^{D-1} (\sin\theta_j)^{j-1} d\theta_j\,, \tag{15}$$

where $r \in [0,\infty)$, $\theta_1 \in [0, 2\pi]$ and $\theta_j \in [0,\pi]$, $j \in [2, D-1]$. The wave function $\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x})$ with a given angular momentum ℓ can be decomposed as a product of a radial wave function $R_\ell(r)$ and the generalized spherical harmonics $Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})$ as[48]

$$\begin{aligned}
&\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = R_\ell(r) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,,\\
&Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) = Y(\ell_1, \ell_2, \ldots, \ell_{D-2}, \ell)\,, \qquad \ell = |m| \text{ for } D = 2\,,
\end{aligned}$$

$$\begin{aligned}
R_\ell(r) &= r^{-(D-1)/2} g(r)\,,\\
Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}} = \theta_1, \theta_2, \ldots, \theta_{D-1}) &= \Phi(\theta_1 = \varphi) H(\theta_2, \ldots, \theta_{D-1})\,,
\end{aligned} \tag{16}$$

which is the simultaneous eigenfunction of L_j^2:

$$\begin{aligned}
L_1^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) &= m^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,,\\
L_j^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) &= \ell_j(\ell_j + j - 1) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,, \quad \ell = 0, 1, \ldots, \ell_k = 0, 1, \ldots, \ell_{k+1},\\
&\qquad j \in [1, D-1], k \in [2, D-2]\,, \quad \ell_1 = -\ell_2, -\ell_2 + 1, \ldots, \ell_2 - 1, \ell_2\,,\\
L_{D-1}^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) &= \ell(\ell + D - 2) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,.
\end{aligned} \tag{17}$$

The unit vector along $\mathbf{x}$ is usually denoted by $\hat{\mathbf{x}} = \mathbf{x}/r$. The substitution of Eqs. (14) and (16) into Eq. (12) allows us to obtain the D-dimensional radial Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)-\frac{L_{D-1}^2}{r^2}+\frac{2\mu}{\hbar^2}\left(E+\frac{a}{r}-\frac{b}{r^2}-c\right)\right\}$$
$$\times\,\psi_{\ell_1\cdots\ell_{D-2}}^{(\ell)}(\mathbf{x})=0\,. \tag{18}$$

The angular momentum operators L_j^2 are defined as:[48–52]

$$L_1^2=-\frac{\partial^2}{\partial\theta_1^2}\,,$$
$$L_k^2=\sum_{a<b=2}^{k+1}L_{ab}^2=-\frac{1}{\sin^{k-1}\theta_k}\frac{\partial}{\partial\theta_k}\left(\sin^{k-1}\theta_k\frac{\partial}{\partial\theta_k}\right)+\frac{L_{k-1}^2}{\sin^2\theta_k},2\le k\le D-1\,,$$
$$L_{ab}=-i\left[x_a\frac{\partial}{\partial x_b}-x_b\frac{\partial}{\partial x_a}\right]. \tag{19}$$

Making use of Eqs. (17) and (19), Eq. (18) leads to the following set of second-order differential equations:

$$\frac{d^2\Phi(\theta_1=\varphi)}{d\theta_1^2}+m^2\Phi(\theta_1=\varphi)=0\,, \tag{20}$$

$$\left[\frac{1}{\sin^{j-1}\theta_j}\frac{d}{d\theta_j}\left(\sin^{j-1}\theta_j\frac{d}{d\theta_j}\right)+\ell_j(\ell_j+j-1)-\frac{\ell_{j-1}(\ell_{j-1}+j-2)}{\sin^2\theta_j}\right]H(\theta_j)$$
$$=0,j\in[2,D-2]\,, \tag{21}$$

$$\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{d}{d\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{d}{d\theta_{D-1}}\right)+\ell(\ell+D-2)-\frac{L_{D-2}^2}{\sin^2\theta_{D-1}}\right.$$
$$\left.-\frac{2\mu\beta}{\hbar^2}\frac{\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right]H(\theta_{D-1})=0\,, \tag{22}$$

$$\left[\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d}{dr}\right)-\frac{\ell(\ell+D-2)}{r^2}\right]R_\ell(r)+\frac{2\mu}{\hbar^2}\left[E-D_e\left(\frac{r-r_e}{r}\right)^2\right]R_\ell(r)=0\,. \tag{23}$$

The solution in Eq. (20) is periodic and must satisfy the period boundary condition $\Phi(\varphi+2\pi)=\Phi(\varphi)$ from which we obtain[45,46]

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}\exp(\pm im\varphi)\,,\qquad m=0,1,2,\ldots\,. \tag{24}$$

Furthermore, Eqs. (21) and (22) representing the angular wave equation take the simple forms

$$\frac{d^2H(\theta_j)}{d\theta_j^2}+(j-1)\frac{\cos\theta_j}{\sin\theta_j}\frac{dH(\theta_j)}{d\theta_j}+\left(\Lambda_j-\frac{\Lambda_{j-1}}{\sin^2\theta_j}\right)H(\theta_j)=0\,, \tag{25}$$

with $j \in [2, D-2]$, $D > 3$ and

$$\frac{d^2 H(\theta_{D-1})}{d\theta_{D-1}^2} + (D-2)\frac{\cos\theta_{D-1}}{\sin\theta_{D-1}}\frac{dH(\theta_{D-1})}{d\theta_{D-1}} + \left[\ell(\ell+D-2) - \frac{\Lambda_{D-2} + (2\mu\beta/\hbar^2)\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right] H(\theta_{D-1}) = 0\,, \quad (26)$$

where $\Lambda_p = \ell_p(\ell_p + p - 1)$, which is well-known in three-dimensional space.[b] Equations (25) and (26) will be solved in the following subsection.

3.2. *The solutions of the D-dimensional angular equations*

In order to apply the NU method, we introduce a new variable, $s = \cos\theta_j$. Hence, Eq. (25) is then rearranged in the form of the universal associated Legendre differential equation:

$$\frac{d^2 H(s)}{ds^2} - \frac{js}{1-s^2}\frac{dH(s)}{ds} + \frac{\Lambda_j - \Lambda_{j-1} - \Lambda_j s^2}{(1-s^2)^2} H(s) = 0\,, \quad (27)$$

where $j \in [2, D-2]$, $D > 3$. By comparing Eqs. (27) and (1), the corresponding polynomials are obtained:

$$\tilde{\tau}(s) = -js\,, \qquad \sigma(s) = 1 - s^2\,, \qquad \tilde{\sigma}(s) = -\Lambda_j s^2 + \Lambda_j - \Lambda_{j-1}\,. \quad (28)$$

Inserting the above expressions into Eq. (9) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(j-2)}{2}s \pm \sqrt{\left[\left(\frac{j-2}{2}\right)^2 + \Lambda_j - k\right] s^2 + k - \Lambda_j + \Lambda_{j-1}}\,. \quad (29)$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} \left(\dfrac{j-2}{2} + \tilde{\Lambda}_{j-1}\right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}\,, \\ \left(\dfrac{j-2}{2} - \tilde{\Lambda}_{j-1}\right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}\,, \\ \dfrac{(j-2)}{2}s + \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\dfrac{j-2}{2}\right)^2, \\ \dfrac{(j-2)}{2}s - \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\dfrac{j-2}{2}\right)^2, \end{cases} \quad (30)$$

[b]$\Lambda_{D-2} = m^2$ for $D = 3$.

where $\tilde{\Lambda}_p = \ell_p + (p-1)/2$, with $p = j-1$, j and $j \in [2, D-2]$, $D > 3$. Imposing the condition $\tau'(s) < 0$ for Eq. (5), one selects the following physically valid solutions:

$$k_1 = \Lambda_j - \Lambda_{j-1} \text{ and } \pi(s) = \left(\frac{j-2}{2} - \tilde{\Lambda}_{j-1}\right) s\,, \tag{31}$$

which yields from Eq. (5) that

$$\tau(s) = -2(1 + \tilde{\Lambda}_{j-1})s\,. \tag{32}$$

Making use of Eqs. (6) and (10), the following expressions for λ are respectively obtained:

$$\lambda = \lambda_n = 2n(1 + \tilde{\Lambda}_{j-1}) + n(n-1)\,, \tag{33}$$

$$\lambda = \Lambda_j - \Lambda_{j-1} - \tilde{\Lambda}_{j-1} + \frac{j-2}{2}\,. \tag{34}$$

Upon comparing Eqs. (33) and (34), we obtain

$$n = \tilde{\Lambda}_j - \tilde{\Lambda}_{j-1} - \frac{1}{2}\,. \tag{35}$$

Furthermore, using Eqs. (2)–(4), and (7) and (8), we obtain the following useful parts of the wave functions:

$$\phi(s) = (1 - s^2)^{\ell_{j-1}/2}\,, \qquad \rho(s) = (1 - s^2)^{\tilde{\Lambda}_{j-1}}\,, \tag{36}$$

where $j \in [2, D-2]$, $D > 3$. Besides, we substitute the weight function $\rho(s)$ given in Eq. (36) into the Rodrigues relation (7) to obtain one of the wave functions in the form

$$y_{n_j}(s) = A_{n_j}(1 - s^2)^{-\tilde{\Lambda}_{j-1}} \frac{d^n}{ds^n}(1 - s^2)^{n + \tilde{\Lambda}_{j-1}}\,, \tag{37}$$

where A_{n_j} is the normalization factor. Finally the angular wave function is

$$H_{n_j}(\theta_j) = N_{n_j}(\sin\theta_j)^{\ell_{j-1}/2} P_{n_j}^{(\tilde{\Lambda}_{j-1}, \tilde{\Lambda}_{j-1})}(\cos\theta_j)\,, \quad j \in [2, D-2]\,, \quad D > 3\,, \tag{38}$$

with

$$n_j = \ell_j - \ell_{j-1}\,, \qquad j \in [2, D-2]\,, \qquad D > 3\,. \tag{39}$$

Likewise, in solving Eq. (26), we introduce a new variable $s = \cos\theta_{D-1}$. Thus, we can also rearrange it as the universal associated Legendre differential equation

$$\frac{d^2H(s)}{ds^2} - \frac{(D-1)s}{1-s^2}\frac{dH(s)}{ds} + \frac{\nu'(1-s^2) - \Lambda'_{D-2}}{(1-s^2)^2} H(s) = 0\,, \tag{40}$$

where

$$\nu' = \ell'(\ell' + D - 2) = \ell(\ell + D - 2) + 2\mu\frac{\beta}{\hbar^2} \text{ and } \Lambda'_{D-2} = \Lambda_{D-2} + 2\mu\frac{\beta}{\hbar^2}\,. \tag{41}$$

Equation (40) has been recently solved in two, three and D dimensions by the NU method in Refs. 46, 54–56. However, the aim in this subsection is to solve it in D

dimensions. Upon letting $D = 3$, we can readily restore 3D solution given in Ref. 46. By comparing Eqs. (40) and (1), the corresponding polynomials are obtained:

$$\tilde{\tau}(s) = -(D-1)s\,, \qquad \sigma(s) = 1 - s^2\,, \qquad \tilde{\sigma}(s) = -\nu' s^2 + \nu' - \Lambda'_{D-2}\,. \tag{42}$$

Inserting the above expressions into Eq. (9) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(D-3)}{2}s \pm \sqrt{\left[\left(\frac{D-3}{2}\right)^2 + \nu' - k\right] s^2 + k - \nu' + \Lambda'_{D-2}}\,. \tag{43}$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} \left(\dfrac{D-3}{2} + \tilde{\Lambda}_{D-2}\right) s & \text{for } k_1 = \nu' - \Lambda'_{D-2}\,, \\ \left(\dfrac{D-3}{2} - \tilde{\Lambda}_{D-2}\right) s & \text{for } k_1 = \nu' - \Lambda'_{D-2}\,, \\ \dfrac{(D-3)}{2}s + \tilde{\Lambda}_{D-2} & \text{for } k_2 = \nu' + \left(\dfrac{D-3}{2}\right)^2\,, \\ \dfrac{(D-3)}{2}s - \tilde{\Lambda}_{D-2} & \text{for } k_2 = \nu' + \left(\dfrac{D-3}{2}\right)^2\,, \end{cases} \tag{44}$$

where $\tilde{\Lambda}_{D-2} = \sqrt{(2\ell_{D-2} + D - 3)^2 + 8\mu\beta/\hbar^2}$. Imposing the condition $\tau'(s) < 0$ for Eq. (5), one selects the following physically valid solutions:

$$k_1 = \nu' - \Lambda'_{D-2} \text{and } \pi(s) = \left(\frac{D-3}{2} - \tilde{\Lambda}_{D-2}\right) s\,, \tag{45}$$

which yields from Eq. (5) that

$$\tau(s) = -2(1 + \tilde{\Lambda}_{D-2})s\,. \tag{46}$$

Making use of Eqs. (6) and (10), the following expressions for λ are respectively obtained:

$$\lambda = \lambda_n = 2n_{D-1}(1 + \tilde{\Lambda}_{D-2}) + n_{D-1}(n_{D-1} - 1)\,, \tag{47}$$

$$\lambda = \nu' - \Lambda'_{D-2} - \tilde{\Lambda}_{D-2} + \frac{D-3}{2}\,. \tag{48}$$

We compare Eqs. (47) and (48), and from the definition $\nu' = \ell'(\ell' + D - 2)$, the new angular momentum ℓ' values are obtained as

$$\ell' = n_{D-1} + \tilde{\Lambda}_{D-2} - \frac{(D-3)}{2}\,, \tag{49}$$

which can be easily reduced to the well-known definition

$$\ell' = \ell = n + m \tag{50}$$

in 3D for the pseudoharmonic potential.[47] Using Eqs. (2)–(4), (7) and (8), we obtain the following useful parts of the wave functions:

$$\phi(s) = (1-s^2)^{(2\tilde{\Lambda}_{D-2}+3-D)/4}\,, \qquad \rho(s) = (1-s^2)^{\tilde{\Lambda}_{D-2}}\,. \tag{51}$$

Besides, we substitute the weight function $\rho(s)$ given in Eq. (51) into the Rodrigues relation (7) and obtain one of the wave functions in the form

$$y_{n_j}(s) = B_{n_j}(1-s^2)^{-\tilde{\Lambda}_{D-2}}\frac{d^{n_j}}{ds^{n_j}}(1-s^2)^{n_j+\tilde{\Lambda}_{D-2}}\,, \qquad j = D-1\,, \tag{52}$$

where $B_{n_{D-1}}$ is the normalization factor. Finally the angular wave function is

$$H_{n_j}(\theta_j) = N_{n_j}(\sin\theta_j)^{\tilde{\Lambda}_{D-2}-(D-3)/2}P_{n_j}^{(\tilde{\Lambda}_{D-2},\tilde{\Lambda}_{D-2})}(\cos\theta_j)\,, \tag{53}$$

where

$$2n_j = \sqrt{(2\ell+D-2)^2+\frac{8\mu\beta}{\hbar^2}} - \sqrt{(2\ell_{D-2}+D-3)^2+\frac{8\mu\beta}{\hbar^2}} - 1\,, \qquad j = D-1\,. \tag{54}$$

Now we are going to solve Eq. (23). After lengthy but straightforward calculations, Eq. (14), representing the radial wave equation can be rewritten as:[6]

$$\frac{d^2g(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E-D_e) + \frac{4\mu D_e r_e}{\hbar^2}\frac{1}{r} - \frac{\tilde{\nu}+(2\mu D_e r_e^2/\hbar^2)}{r^2}\right]g(r) = 0\,, \tag{55}$$

where

$$\tilde{\nu} = \frac{1}{4}(M-1)(M-3)\,, \qquad M = D+2\ell\,. \tag{56}$$

Obviously, the two particles in Eq. (55) interacting via Coulombic-like field appear to have a slight additional change in the angular momentum barrier which acts as a repulsive core, which for any arbitrary ℓ, prevents collapse of the system in any space dimension due to the additional centrifugal potential barrier.

3.3. *The solutions of the radial equation*

The solution of the SE for the modified central Kratzer's potential has already been solved by means of the NU method in Ref. 47. Very recently, using the same method, the problem for the non-central potential in Eq. (11) has been solved in 3D by Cheng and Dai.[46] However, the aim of this subsection is to solve the problem with a different radial separation function $g(r)$ in any arbitrary dimensions. We now study the bound state (real) solutions (i.e. $E < D_e$) of Eq. (55). Letting

$$\varepsilon = \sqrt{-\frac{2\mu}{\hbar^2}(E-D_e)}\,, \qquad \alpha = \frac{4\mu D_e r_e}{\hbar^2}\,, \qquad \gamma = \tilde{\nu} + \frac{1}{2}\alpha r_e\,, \tag{57}$$

and substituting these expressions in Eq. (55), one gets

$$\frac{d^2g(r)}{dr^2} + \left(\frac{-\varepsilon^2r^2+\alpha r-\gamma}{r^2}\right)g(r) = 0\,. \tag{58}$$

To apply the conventional NU method, Eq. (58) is compared with Eq. (1) and the following expressions are obtained:

$$\tilde{\tau}(r) = 0\,, \qquad \sigma(r) = r\,, \qquad \tilde{\sigma}(r) = -\varepsilon^2 r^2 + \alpha r - \gamma\,. \tag{59}$$

Substituting the above expressions into Eq. (9) gives

$$\pi(r) = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\varepsilon^2 r^2 + 4(k-\alpha)r + 4\gamma + 1}\,. \tag{60}$$

According to this conventional method, the expression in the square root is the square of a polynomial. Thus, the two roots k can be readily obtained as

$$k = \alpha \pm \varepsilon\sqrt{4\gamma + 1}\,. \tag{61}$$

In view of that, we arrive at the following four possible functions of $\pi(r)$:

$$\pi(r) = \begin{cases} \dfrac{1}{2} + \left[\varepsilon r + \dfrac{1}{2}\sqrt{4\gamma+1}\right] & \text{for } k_1 = \alpha + \varepsilon\sqrt{4\gamma+1}\,, \\ \dfrac{1}{2} - \left[\varepsilon r + \dfrac{1}{2}\sqrt{4\gamma+1}\right] & \text{for } k_1 = \alpha + \varepsilon\sqrt{4\gamma+1}\,, \\ \dfrac{1}{2} + \left[\varepsilon r - \dfrac{1}{2}\sqrt{4\gamma+1}\right] & \text{for } k_2 = \alpha - \varepsilon\sqrt{4\gamma+1}\,, \\ \dfrac{1}{2} - \left[\varepsilon r - \dfrac{1}{2}\sqrt{4\gamma+1}\right] & \text{for } k_2 = \alpha - \varepsilon\sqrt{4\gamma+1}\,. \end{cases} \tag{62}$$

The correct value of $\pi(r)$ is chosen such that the function $\tau(r)$ given by Eq. (5) will have negative derivative.[12] So we can select the physical values to be

$$k = \alpha - \varepsilon\sqrt{4\gamma+1} \text{ and } \pi(r) = \frac{1}{2} - \left[\varepsilon r - \frac{1}{2}\sqrt{4\gamma+1}\right]\,, \tag{63}$$

which yield

$$\tau(r) = -2\varepsilon r + (1 + \sqrt{4\gamma+1})\,. \tag{64}$$

Upon using Eqs. (6) and (10), the following expressions for λ are respectively obtained:

$$\lambda = \lambda_n = 2N\varepsilon\,, \qquad N = 0, 1, 2, \ldots\,, \tag{65}$$

$$\lambda = \alpha - \varepsilon(1 + \sqrt{4\gamma+1})\,. \tag{66}$$

So we can obtain the energy eigenvalues as

$$E_N = D_e - \frac{8\mu D_e^2 r_e^2/\hbar^2}{(2N + 1 + \sqrt{(M-1)(M-3) + 8\mu D_e r_e^2/\hbar^2 + 1})^2}\,, \tag{67}$$

where

$$(M-1)(M-3) = 4\tilde{\nu} = (2\ell' + D - 2)^2 - \frac{8\mu\beta}{\hbar^2 - 1}\,, \tag{68}$$

with ℓ' defined in Eq. (49). Therefore, the final energy spectra in Eq. (67) takes the following Coulombic-like form:[7]

$$E_{N'} = D_e - \frac{2\mu D_e^2 r_e^2/\hbar^2}{(N')^2}\,, \qquad N' = 0, 1, 2, \dots\,, \tag{69}$$

where

$$N' = \frac{1}{2}\left[2N + 1 + \sqrt{(2\ell' + D - 2)^2 + \frac{8\mu(D_e r_e^2 - \beta)}{\hbar^2}}\,\right] = N + L + 1\,, \tag{70}$$

is simply obtained by means of substituting Eq. (68) into Eq. (67).

(i) Equation (69), with the help of Eq. (49), is transformed into the following general form:

$$E_N = D_e - \frac{8\mu D_e^2 r_e^2/\hbar^2}{\left(2N + 1 + \sqrt{2n_{D-1} + 1 + \sqrt{(2\ell_{D-2} + D - 3)^2 + 8\mu\beta/\hbar^2})^2 + 8\mu(D_e r_e^2 - \beta)/\hbar^2}\right)^2}\,, \tag{71}$$

and it is consistent with Eq. (40) in Ref. 46.

(ii) If $D = 3$ and $\beta = 0$ (modified Kratzer potential), Eq. (71) is transformed into the form

$$E_n = D_e - \frac{8\mu D_e^2 r_e^2/\hbar^2}{(1 + 2N + \sqrt{(2\ell + 1)^2 + 8\mu D_e r_e^2/\hbar^2})^2}\,, \tag{72}$$

and it is consistent with Eq. (14) in Ref. 47.

Let us now turn our attention to find the radial wave functions for this potential. Using $\tau(r)$, $\pi(r)$ and $\sigma(r)$ in Eqs. (4) and (8), we find

$$\phi(r) = r^{(\sqrt{4\gamma+1}+1)/2} e^{-\varepsilon r}\,, \tag{73}$$

$$\rho(r) = r^{\sqrt{4\gamma+1}} e^{-2\varepsilon r}\,. \tag{74}$$

Then from Eq. (7), we obtain

$$y_n(r) = B_n r^{-\sqrt{4\gamma+1}} e^{2\varepsilon r} \frac{d^N}{dr^N}(r^{N+\sqrt{4\gamma+1}} e^{-2\varepsilon r})\,, \tag{75}$$

and the wave function $g(r)$ can be written in the form of the generalized Laguerre polynomials as

$$g(r) = C_{N,L} r^{L+1} e^{-\varepsilon r} L_N^{2L+1}(2\varepsilon r)\,, \tag{76}$$

where L can be found easily by comparing the two sides of Eq. (70) to be $L = \sqrt{(2\ell' + D - 2)^2 + 8\mu(D_e r_e^2 - \beta)/\hbar^2} - 1$. Finally, the radial wave functions of the Schrödinger equation are obtained:

$$R(r) = C_{N,L} r^{L-(D-3)/2} e^{-\varepsilon r} L_N^{2L+1}(2\varepsilon r)\,, \tag{77}$$

where

$$\varepsilon = \frac{2\mu D_e r_e}{\hbar^2 N'}\,, \tag{78}$$

with N' as given in Eq. (70) and $C_{N,L}$ is the normalization constant to be determined below. Using the normalization condition, $\int_0^\infty R^2(r)r^{D-1}dr = 1$, and the orthogonality relation of the generalized Laguerre polynomials, $\int_0^\infty z^{\eta+1}e^{-z}[L_n^\eta(z)]^2 dz = (2n+\eta+1)(n+\eta)!/n!$, we have

$$C_{N,L} = \sqrt{\frac{(2\varepsilon)^{2L+3}N!}{2(N+L+1)(N+2L+1)!}}\,. \tag{79}$$

Therefore, we may express the un-normalized total wave functions as

$$\begin{aligned}\psi(\mathbf{x}) = &\sqrt{\frac{(2\varepsilon)^{2L+3}N!}{4\pi(N+L+1)(N+2L+1)!}}r^{L-(D-3)/2}\exp(-\varepsilon r)L_N^{2L+1}(2\varepsilon r)\\ &\times \exp(\pm im\varphi)\prod_{j=2}^{D-2} N_{n_j}(\sin\theta_j)^{\ell_{j-1}}P_{n_j}^{(\tilde{\Lambda}_{j-1},\tilde{\Lambda}_{j-1})}(\cos\theta_j)\\ &\times N_{n_{D-1}}\sin(\theta_{D-1})^{\tilde{\Lambda}_{D-2}-(D-3)/2}P_{n_{D-1}}^{(\tilde{\Lambda}_{D-2},\tilde{\Lambda}_{D-2})}(\cos\theta_{D-1})\,.\end{aligned} \tag{80}$$

4. Results and Conclusions

In this work, we have solved the Schrödinger equation in any arbitrary dimension for its exact bound-states with a recently proposed modified Kratzer plus ring-shaped potential by means of a simple NU method. The analytical expressions for the total energy levels of this system is found to be different from the results obtained for the modified Kratzer's potential in Ref. 47 and is also more general than the one obtained recently in 3D[46] (e.g. cf. Ref. 54). Therefore, the non-central potentials treated in Ref. 44 can be introduced as perturbation to the modified Kratzer's potential by adjusting the strength of the coupling constant β in terms of D_e, which is the coupling constant of the modified Kratzer's potential. In addition, the angular part, the radial part, and then the total wave functions, are also found. Thus, the Schrödinger equation with a new non-central but separable potential has also been studied (cf. Ref. 44 and references therein). This method is very simple and useful in solving other complicated systems analytically without given restriction conditions on the solution of some quantum systems as is the case in the other models. Finally, we point out that these exact results obtained for this new proposed form of the potential (11) may have some interesting applications in the study of different quantum mechanical systems, atomic and molecular physics.

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