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EXACT BOUND STATES OF THE D-DIMENSIONAL KLEIN–GORDON EQUATION WITH EQUAL SCALAR AND VECTOR RING-SHAPED PSEUDOHARMONIC POTENTIAL

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We present the exact solution of the Klein–Gordon equation in D-dimensions in the presence of the equal scalar and vector pseudoharmonic potential plus the ring-shaped potential using the Nikiforov–Uvarov method. We obtain the exact bound state energy levels and the corresponding eigen functions for a spin-zero particles. We also find that the solution for this ring-shaped pseudoharmonic potential can be reduced to the three-dimensional ($3D$) pseudoharmonic solution once the coupling constant of the angular part of the potential becomes zero.

Keywords: Bound states; energy eigenvalues and eigenfunctions; Klein–Gordon equation; pseudoharmonic potential; ring-shaped potential; non-central potentials; Nikiforov and Uvarov method.

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1. Introduction

In nuclear and high energy physics,[1,2] one of the interesting problems is to obtain exact solutions of the relativistic wave equations like Klein–Gordon, Dirac and Salpeter wave equations for mixed vector and scalar potential. The Klein–Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. The Klein–Gordon equation has also been used to understand the motion of a spin-0 particle in large class of potentials. In recent years, much works have been done to solve these relativistic wave equations for various potentials by using different methods. These relativistic equations contain two objects: the four-vector linear momentum operator and the scalar rest mass.

They allow us to introduce two types of potential coupling, which are the four-vector potential (V) and the space-time scalar potential (S).

For the case $S = \pm V$, the solution of these wave equations with physical potentials has been studied recently. The exact solutions of these equations are possible only for certain central potentials such as Morse potential,[3] Hulthén potential,[4] Woods–Saxon potential,[5] Pöschl–Teller potential,[6] reflectionless-type potential,[7] pseudoharmonic oscillator,[8] ring-shaped harmonic oscillator,[9] $V_0 \tanh^2(r/r_0)$ potential,[10] five-parameter exponential potential,[11] Rosen–Morse potential,[12] and generalized symmetrical double-well potential,[13] etc. by using different methods. It is remarkable that in most works in this area, the scalar and vector potentials are almost taken to be equal (i.e., $S = V$).[2,14] However, in some other cases, it is considered to be the case where the scalar potential is greater than the vector potential (in order to guarantee the existence of Klein–Gordon bound states) (i.e., $S > V$).[15–19] Nonetheless, such physical potentials are very few. The bound state solutions for the last case is obtained for the exponential potential for the s-wave Klein–Gordon equation when the scalar potential is greater than the vector potential.[15]

On the other hand, the other exactly solvable potentials are the ring-shaped potentials.[20] These potentials involve an attractive Coulomb potential plus a repulsive inverse square potential, that is, one like the Coulombic ring-shaped potential[21,22] revived in quantum chemistry by Hartmann *et al.*[23] The oscillatory ring-shaped potential studied by Quesne[24] have been investigated using various quantum mechanical approaches.[25] In taking the relativistic effects into account for spin-0 particle in the presence of a class of angular dependent potentials, Ikhdair and Sever[26] applied the Nikiforov–Uvarov method[27] in solving the Klein–Gordon equation in D-dimensions to a spin-zero particle for the case $V = S$ ring-shaped Kratzer type potential.[21] Furthermore, Berkdemir[28] also applied the same method to solve the Klein–Gordon equation of a spin-zero particle subjects to a Kratzer-type potential.

Recently, Chen and Dong[29] proposed a ring-shaped potential and obtained the exact solution of the Schrödinger equation for the Coulomb potential plus this ring-shaped potential which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene. This type of potential used by Ref. 29 appears to be very similar to the potential used by Ref. 26. Additionally, Cheng and Dai,[30] proposed another potential consisting of the modified Kratzer's potential[31] plus the proposed ring-shaped potential in Ref. 29. They have presented the energy eigenvalues for this proposed exactly-solvable potential in $3D$ Schrödinger equation using the NU method. The two quantum systems solved by Refs. 29 nad 30 are closely relevant to each other as they deal with a Coulombic field interaction except for an additional change in the angular momentum barrier. In addition, the D-dimensional Schrödinger and Klein–Gordon wave equations have been solved for some types of angular and radial dependent potentials using the NU method.[22,25,26,32]

The aim of the present paper is to obtain the exact bound state solutions of the D-dimensional Klein–Gordon with an oscillatory-type plus a ring-shaped potential.

The radial and angular parts of the Klein–Gordon equation with this type of potential are solved using the NU method.

This work is organized as follows: in Sec. 2, we shall present the Klein–Gordon equation in spherical coordinates for spin-0 particle with an equal scalar and vector oscillatory-type ring-shaped potential. We separate the wave equation into radial and angular parts. Section 3 is devoted to a brief description of the NU method. In Sec. 4, we present the exact bound state solutions to the radial and angular equations in D-dimensions. Finally, the relevant conclusions are given in Sec. 5.

2. The Klein–Gordon Equation with Equal Scalar and Vector Potentials

In relativistic quantum mechanics, we usually use the Klein–Gordon equation for describing a scalar particle, i.e., the spin-0 particle dynamics. The discussion of the relativistic behavior of spin-zero particles requires understanding the single particle spectrum and the exact solutions to the Klein–Gordon equation which are constructed by using the four-vector potential $\mathbf{A}_\lambda$ ($\lambda = 0, 1, 2, 3$) and the scalar potential (S). In order to simplify the solution of the Klein–Gordon equation, the four-vector potential can be written as $\mathbf{A}_\lambda = (A_0, 0, 0, 0)$. The first component of the four-vector potential is represented by a vector potential (V), i.e., $A_0 = V$. In this case, the motion of a relativistic spin-0 particle in a potential is described by the Klein–Gordon equation with the potentials V and S.[1] For the case $S \geq V$, there exist bound state (real) solutions for a relativistic spin-0 particle.[15–19] On the other hand, for $S = V$, the Klein–Gordon equation reduces to a Schrödinger-like equation and thereby the bound state solutions are easily obtained by using the well-known methods developed in nonrelativistic quantum mechanics.[2]

The Klein–Gordon equation describing a scalar particle (spin-0 particle) with scalar $S(r, \theta, \varphi)$ and vector $V(r, \theta, \varphi)$ potentials is given by Refs. 2 and 14

$$\{\mathbf{P}^2 - [V(r, \theta, \varphi) - E_R]^2 + [S(r, \theta, \varphi) + \mu]^2\}\psi(r, \theta, \varphi) = 0\,, \tag{1}$$

where μ is the rest mass, E_R is the relativistic energy, $\mathbf{P}$ is the momentum operator and S and V are the scalar and vectorial potentials. Alhaidari *et al.*[14] concluded that only the choice $S = V$ produces a non-trivial non-relativistic limit with a potential function $2V$, not V. Accordingly, it would be natural to scale the potential terms in Eq. (1) so that in the non-relativistic limit the interaction potential becomes V, not $2V$.

Therefore, they modified Eq. (1) to read as follows (in the relativistic natural units $\hbar = c = 1$):

$$\left\{\nabla^2 + \left[\frac{1}{2}V(r, \theta, \varphi) - E_R\right]^2 - \left[\frac{1}{2}S(r, \theta, \varphi) + \mu\right]^2\right\}\psi(r, \theta, \varphi) = 0\,. \tag{2}$$

After substituting $S(r,\theta,\varphi) = V(r,\theta,\varphi)$, the equal scalar and vector potentials case, Eq. (2) becomes

$$\{\nabla^2 - (E_R+\mu)V(r,\theta,\varphi) + E_R^2 - \mu^2\}\psi(r,\theta,\varphi) = 0\,. \tag{3}$$

Moreover, if we take the interaction potential in Eq. (3) as a general oscillatory-type ring-shaped potential, the D-dimensional Klein–Gordon equation is separated into variables and the resulting equation can be solved through the NU method.

We take the interaction potential in Eq. (3) to be of an oscillatory-type plus a ring-shaped potential which is the potential of a diatomic molecule:[25]

$$\begin{aligned} V(r,\theta,\varphi) &= V_1(r) + \frac{V_2(\theta)}{r^2} + \frac{V_3(\varphi)}{r^2\sin^2\theta}\,, \\ V_1(r) &= Ar^2 + \frac{B}{r^2} + C, V_2(\theta) = \beta ctg^2\theta, V_3(\varphi) = 0\,, \end{aligned} \tag{4}$$

where $A = a_0 r_0^{-2}$, $B = a_0 r_0^2$, $C = -2a_0$ and β is a positive real constant, a_0 is the dissociation energy and r_0 is the equilibrium internuclear distance.[33,34] The potentials in Eq. (4) can be reduced to pseudoharmonic potential in the limiting case of $\beta = 0$.[33,34] Nonetheless, the energy spectrum for this potential can be obtained directly by considering it as a special case of the general angular dependent seperable potentials.[20]

Our aim is to derive analytically the exact energy spectrum for a moving particle in the presence of the potential given by Eq. (4) in a very simple way. We begin by considering the Schrödinger equation in an arbitrary dimension D for our proposed potential[25,26]

$$\begin{aligned} &\left\{\nabla_D^2 + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{1}{r^2}V(\theta)\right]\right\}\psi_{\ell_1\cdots\ell_{D-2}}^{(\ell_{D-1}=\ell)}(\mathbf{x}) = 0\,, \\ &\nabla_D^2 = \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r}\frac{\partial}{\partial r} \\ &\qquad + \frac{1}{r^2}\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{\partial}{\partial\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{\partial}{\partial\theta_{D-1}}\right) - \frac{L_{D-2}^2}{\sin^2\theta_{D-1}}\right]\,, \\ &\psi_{\ell_1\cdots\ell_{D-2}}^{(\ell)}(\mathbf{x}) = R_\ell(r)Y_{\ell_1\cdots\ell_{D-2}}^{(\ell)}(\hat{\mathbf{x}}), R_\ell(r) = r^{-(D-1)/2}g(r)\,, \end{aligned} \tag{5}$$

where μ and E denote the reduced mass and energy of two interacting particles, respectively. $\mathbf{x}$ is a D-dimensional position vector with the hyperspherical Cartesian components $x_1, x_2, \ldots, x_D$ given as follows:[a,35–40]

$$\begin{aligned} x_1 &= r\cos\theta_1\sin\theta_2\cdots\sin\theta_{D-1}\,, \\ x_2 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-1}\,, \end{aligned}$$

[a]It is worth noting that such a definition was introduced by Erdélyi early in 1950s (cf., Ref. 40, pp. 232–235, Chapter 11) even though the notation used by him is quite different from that by Louck and Chatterjee.

$$
\begin{gathered}
x_3 = r\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{D-1}\,, \\
\vdots \\
x_j = r\cos\theta_{j-1}\sin\theta_j \cdots \sin\theta_{D-1}, 3 \le j \le D-1\,, \\
\vdots \\
x_{D-1} = r\cos\theta_{D-2}\sin\theta_{D-1}\,, \\
x_D = r\cos\theta_{D-1}, \sum_{j=1}^{D} x_j^2 = r^2\,,
\end{gathered}
\tag{6}
$$

for $D = 2, 3, \ldots$. We have $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ for $D = 2$ and $x_1 = r\cos\varphi\sin\theta$, $x_2 = r\sin\varphi\sin\theta$, $x_3 = r\cos\theta$ for $D = 3$. The Laplace operator ∇_D^2 is defined by

$$
\nabla_D^2 = \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2}\,. \tag{7}
$$

The volume element of the configuration space is given by

$$
\prod_{j=1}^{D} dx_j = r^{D-1} dr d\Omega, d\Omega = \prod_{j=1}^{D-1} (\sin\theta_j)^{j-1} d\theta_j\,, \tag{8}
$$

where $r \in [0, \infty)$, $\theta_1 \in [0, 2\pi]$ and $\theta_j \in [0, \pi]$, $j \in [2, D-1]$. The wave function $\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x})$ with a given angular momentum ℓ can be decomposed as a product of a radial wave function $R_\ell(r)$ and the generalized angular dependent spherical harmonics $Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})$ as[35]

$$
\begin{gathered}
\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = R_\ell(r) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}), \\
Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) = Y(\ell_1, \ell_2, \ldots, \ell_{D-2}, \ell), \ell = |m| \text{ for } D = 2\,, \\
R_\ell(r) = r^{-(D-1)/2} g(r)\,, \\
Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}} = \theta_1, \theta_2, \ldots, \theta_{D-1}) = \Phi(\theta_1 = \varphi) H(\theta_2, \ldots, \theta_{D-1})\,,
\end{gathered}
\tag{9}
$$

which is the simultaneous eigenfunction of L_j^2:

$$
\begin{gathered}
L_1^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) = m^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,, \\
L_j^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) = \ell_j(\ell_j + j - 1) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,, \\
\ell = 0, 1, \ldots, \ell_k = 0, 1, \ldots, \ell_{k+1}, j \in [1, D-1], k \in [2, D-2]\,, \\
\ell_1 = -\ell_2, -\ell_2 + 1, \ldots, \ell_2 - 1, \ell_2\,, \\
L_{D-1}^2 Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}}) = \ell(\ell + D - 2) Y^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\hat{\mathbf{x}})\,,
\end{gathered}
\tag{10}
$$

where the unit vector along $\mathbf{x}$ is usually denoted by $\hat{\mathbf{x}} = \mathbf{x}/r$.

Hence for a nonrelativistic treatment with the same potential, the Schrödinger equation in spherical coordinates is

$$\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)-\frac{\ell_{D-1}(\ell_{D-1}+D-2)}{r^2}\right.$$
$$\left.+\frac{2\mu}{\hbar^2}\left(E_{NR}-V_1(r)-\frac{V_2(\theta)}{r^2}-\frac{V_3(\varphi)}{r^2\sin^2\theta}\right)\right\}R_\ell(r)=0\,, \tag{11}$$

where μ and E_{NR} are the reduced mass and the nonrelativistic energy, respectively. The angular momentum operators L_j^2 are defined as:[35–39]

$$L_1^2=-\frac{\partial^2}{\partial\theta_1^2}\,,$$
$$L_k^2=\sum_{a<b=2}^{k+1}L_{ab}^2=-\frac{1}{\sin^{k-1}\theta_k}\frac{\partial}{\partial\theta_k}\left(\sin^{k-1}\theta_k\frac{\partial}{\partial\theta_k}\right)+\frac{L_{k-1}^2}{\sin^2\theta_k},2\le k\le D-1\,,$$
$$L_{ab}=-i\left[x_a\frac{\partial}{\partial x_b}-x_b\frac{\partial}{\partial x_a}\right]\,. \tag{12}$$

Making use of Eqs. (10) and (12), leads to the separation of Eq. (11) into the following set of second-order differential equations:

$$\frac{d^2\Phi(\theta_1=\varphi)}{d\theta_1^2}+m^2\Phi(\theta_1=\varphi)=0\,, \tag{13}$$

$$\left[\frac{1}{\sin^{j-1}\theta_j}\frac{d}{d\theta_j}\left(\sin^{j-1}\theta_j\frac{d}{d\theta_j}\right)+\ell_j(\ell_j+j-1)\right.$$
$$\left.-\frac{\ell_{j-1}(\ell_{j-1}+j-2)}{\sin^2\theta_j}\right]H(\theta_j)=0,j\in[2,D-2]\,, \tag{14}$$

$$\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{d}{d\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{d}{d\theta_{D-1}}\right)+\lambda_\ell\right.$$
$$\left.-\frac{1}{\sin^2\theta_{D-1}}\left(L_{D-2}^2+\frac{2\mu C}{\hbar^2}\cos^2\theta_{D-1}\right)\right]H(\theta_{D-1})=0\,, \tag{15}$$

$$\left[\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d}{dr}\right)-\frac{\lambda_\ell}{r^2}\right]R_\ell(r)+\frac{2\mu}{\hbar^2}\left[E_{NR}+\frac{A}{r}-\frac{B}{r^2}\right]R(r)=0\,, \tag{16}$$

with m^2 and $\lambda_\ell=\ell(\ell+D-2)$ two separation constants whereas μ and E_{NR} are the reduced mass and the nonrelativistic energy, respectively.

On the other hand, the D-dimensional Klein–Gordon equation in Eq. (1) (we shall pass to natural units with $\hbar = c = 1$) becomes[28]

$$\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\tilde{\ell}(\tilde{\ell}+D-2)}{r^2} - \left(E_R+\mu\right)\left(V_1(r)+\frac{V_2(\theta)}{r^2}\right) + (E_R^2-\mu^2)\right\}\psi_{n\tilde{\ell}m}(r,\theta,\varphi) = 0\,. \tag{17}$$

With the total wave function being the same representation as in Eq. (9) and with the transformation $\ell \to \tilde{\ell}$,

$$\psi_{n\tilde{\ell}m}(r,\theta,\varphi) = R_{\tilde{\ell}}(r)Y_{\tilde{\ell}}^m(\theta,\varphi)\,, \qquad R_{\tilde{\ell}}(r) = r^{-(D-1)/2}g(r)\,,$$
$$Y_{\tilde{\ell}}^m(\theta,\varphi) = \prod_{j=2}^{D-1} H_j(\theta)\Phi(\varphi)\,, \tag{18}$$

and employing the method of the separation of variables leads to the following differential equations:[20,25,26,41,42]

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + \tilde{\ell}_1^2\Phi(\varphi) = 0, \tilde{\ell}_1 = m\,, \tag{19}$$

$$\left[\frac{1}{\sin^{j-1}\theta_j}\frac{d}{d\theta_j}\left(\sin^{j-1}\theta_j\frac{d}{d\theta_j}\right) + \tilde{\ell}_j(\tilde{\ell}_j+j-1) - \frac{\tilde{\ell}_{j-1}(\tilde{\ell}_{j-1}+j-2)}{\sin^2\theta_j}\right]H(\theta_j) = 0, j \in [2, D-2]\,, \tag{20}$$

$$\left[\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{d}{d\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{d}{d\theta_{D-1}}\right) + \lambda_{\tilde{\ell}} - \frac{\tilde{L}_{D-2}^2 + C\alpha_2^2\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right]H(\theta_{D-1}) = 0\,, \tag{21}$$

$$\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{dR(r)}{dr}\right) - \left[\frac{\lambda_{\tilde{\ell}}}{r^2} + \alpha_2^2\left(\alpha_1^2 - \frac{A}{r} + \frac{B}{r^2}\right)\right]R_{\tilde{\ell}}(r) = 0\,, \tag{22}$$

where $\alpha_1^2 = \mu - E_R$, $\alpha_2^2 = \mu + E_R$, m and $\tilde{\ell}$ are constants with m^2 and $\lambda_{\tilde{\ell}} = \tilde{\ell}(\tilde{\ell}+D-2)$ are the separation constants.

Equations (19)–(22) have the same functional form as Eqs. (13)–(16). Therefore, the solution of the Klein–Gordon equation can be reduced to the solution of the Schrödinger equation with the appropriate choice of parameters: $\tilde{\ell} \to \ell$, $\alpha_1^2 \to -E_{NR}$ and $\alpha_2^2 \to 2\mu/\hbar^2$.

The solution of Eq. (19) is well-known periodic and must satisfy the period boundary condition $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ which is the azimuthal angle solution:

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(\pm i\tilde{\ell}_1\varphi), \tilde{\ell}_1 = m = 0, 1, 2, \ldots . \tag{23}$$

It is worth to note that the Eqs. (20)–(22), the polar angle and radial equations, are to be solved by using NU method[27] which is given briefly in the following section.

3. Nikiforov–Uvarov Method

The NU method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type.[20,25–28,42–45] In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{24}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. Using a wave function, $\psi_n(s)$, of the simple form

$$\psi_n(s) = \phi_n(s)y_n(s)\,, \tag{25}$$

reduces Eq. (24) into an equation of a hypergeometric type

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0\,, \tag{26}$$

where

$$\sigma(s) = \pi(s)\frac{\phi(s)}{\phi'(s)}\,, \tag{27}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tau'(s) < 0\,, \tag{28}$$

and λ is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s)\,, \qquad n = 0, 1, 2, \ldots . \tag{29}$$

The polynomial $\tau(s)$ with the parameter s and prime factors shows the differentials at first degree are negative. It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. Furthermore, the other part $y_n(s)$ of the wave function Eq. (25) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)]\,, \tag{30}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition[27]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s)\,, \qquad w(s) = \sigma(s)\rho(s)\,. \tag{31}$$

The function π and the parameter λ are defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}\,, \tag{32}$$

$$\lambda = k + \pi'(s)\,. \tag{33}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (32) can be arranged to be the square of a polynomial of first degree.[27] This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are substituted in Eq. (32). In addition, by comparing Eqs. (29) and (33), we obtain the energy eigenvalues.

4. Exact Solutions of the Radial and Angle-Dependent Equations

4.1. *The solutions of the D-dimensional angular equations*

At the beginning, we rewrite Eqs. (20) and (21) representing the angular wave equations in the following simple forms:[42]

$$\frac{d^2H(\theta_j)}{d\theta_j^2} + (j-1)ctg\theta_j \frac{dH(\theta_j)}{d\theta_j} + \left(\Lambda_j - \frac{\Lambda_{j-1}}{\sin^2\theta_j}\right) H(\theta_j) = 0\,, \quad j \in [2, D-2], D > 3\,, \tag{34}$$

$$\frac{d^2H(\theta_{D-1})}{d\theta_{D-1}^2} + (D-2)ctg\theta_{D-1}\frac{dH(\theta_{D-1})}{d\theta_{D-1}} + \left[\tilde{\ell}(\tilde{\ell} + D - 2) - \frac{\Lambda_{D-2} + \beta\alpha_2^2\cos^2\theta_{D-1}}{\sin^2\theta_{D-1}}\right] H(\theta_{D-1}) = 0\,, \tag{35}$$

where $\Lambda_p = \tilde{\ell}_p(\tilde{\ell}_p + p - 1)$, $p = j-1$, j with $\tilde{\ell}_p = 0, 1, 2, \ldots$ is the angular quantum numbers in D-dimensions which is well-known in $3D$ space.[b] Equations (34) and (35) will be solved in the following subsection. Employing $s = \cos\theta_j$, we transform Eq. (34) to the associated-Legendre equation

$$\frac{d^2H(s)}{ds^2} - \frac{js}{1-s^2}\frac{dH(s)}{ds} + \frac{\Lambda_j - \Lambda_{j-1} - \Lambda_j s^2}{(1-s^2)^2} H(s) = 0\,, \quad j \in [2, D-2], D > 3\,. \tag{36}$$

By comparing Eqs. (36) and (24), the corresponding polynomials are obtained

$$\tilde{\tau}(s) = -js\,, \qquad \sigma(s) = 1 - s^2\,, \qquad \tilde{\sigma}(s) = -\Lambda_j s^2 + \Lambda_j - \Lambda_{j-1}\,. \tag{37}$$

Inserting the above expressions into Eq. (32) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(j-2)}{2}s \pm \sqrt{\left[\left(\frac{j-2}{2}\right)^2 + \Lambda_j - k\right] s^2 + k - \Lambda_j + \Lambda_{j-1}}\,. \tag{38}$$

[b] $\Lambda_{D-2} = m^2$ for $D = 3$.

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} \left(\dfrac{j-2}{2} + \tilde{\Lambda}_{j-1}\right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}\,, \\ \left(\dfrac{j-2}{2} - \tilde{\Lambda}_{j-1}\right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}\,, \\ \dfrac{(j-2)}{2} s + \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\dfrac{j-2}{2}\right)^2\,, \\ \dfrac{(j-2)}{2} s - \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\dfrac{j-2}{2}\right)^2\,, \end{cases} \tag{39}$$

where $\tilde{\Lambda}_p = \tilde{\ell}_p + (p-1)/2$, with $p = j-1$, $j \in [2, D-2]$, $D > 3$. Imposing the condition $\tau'(s) < 0$ for Eq. (28), one selects the following physically valid solutions with $\tau' = \tau'(\tilde{\ell}_{j-1})$; that is, a function of the angular momentum:

$$k_1 = \Lambda_j - \Lambda_{j-1} \quad \text{and} \quad \pi(s) = -\tilde{\ell}_{j-1} s\,, \qquad j \in [2, D-2], D > 3\,. \tag{40}$$

This condition leads to

$$\tau(s) = -2(1 + \tilde{\Lambda}_{j-1}) s\,. \tag{41}$$

Making use of Eqs. (29) and (33), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_{n_j} = 2n_j(1 + \tilde{\Lambda}_{j-1}) + n_j(n_j - 1)\,, \tag{42}$$

$$\lambda = \Lambda_j - \Lambda_{j-1} - \tilde{\Lambda}_{j-1} + \frac{j-2}{2}\,. \tag{43}$$

Upon comparing Eqs. (42) and (43), we obtain

$$n_j = \tilde{\ell}_j - \tilde{\ell}_{j-1}\,. \tag{44}$$

Furthermore, Eqs. (25)–(27) and (30)–(31), give

$$\phi(s) = (1 - s^2)^{\tilde{\ell}_{j-1}/2}, \rho(s) = (1 - s^2)^{\tilde{\ell}_{j-1} + (j-2)/2}\,. \tag{45}$$

We substitute the wight function $\rho(s)$ given in Eq. (45) into the Rodrigues relation Eq. (30) to obtain the following wavefunction:

$$y_{n_j}(s) = A_{n_j}(1 - s^2)^{-\tilde{\Lambda}_{j-1}} \frac{d^{n_j}}{ds^{n_j}} (1 - s^2)^{n_j + \tilde{\Lambda}_{j-1}}\,, \tag{46}$$

where A_{n_j} is the normaliation factor. Finally the angular wavefunction is

$$H_{n_j}(\theta_j) = N_{n_j} (\sin\theta_j)^{\tilde{\ell}_j - n_j} P_{n_j}^{(\tilde{\ell}_j - n_j + (j-2)/2, \tilde{\ell}_j - n_j + (j-2)/2)}(\cos\theta_j)\,, \tag{47}$$

where the quantum numbers n_j are defined in Eq. (44) and the normalization factor is

$$N_{n_j} = \sqrt{\frac{(2\ell' + 1)\Gamma(\ell' - m')}{2\Gamma(\ell' + m')}} = \sqrt{\frac{(2\tilde{\ell}_j + j - 1) n_j!}{2\Gamma(\tilde{\ell}_j + \tilde{\ell}_{j-1} + j - 2)}}\,, \quad j \in [2, D-2]\,, \quad D > 3\,. \tag{48}$$

Likewise, in solving Eq. (35), we introduce a new variable $s = \cos\theta_{D-1}$. Thus, we can also rearrange it as the universal associated-Legendre differential equation

$$\frac{d^2H(s)}{ds^2} - \frac{(D-1)s}{1-s^2}\frac{dH(s)}{ds} + \frac{\nu'(1-s^2) - \Lambda'_{D-2}}{(1-s^2)^2}H(s) = 0\,, \tag{49}$$

where

$$\nu' = \ell'(\ell' + D - 2) = \tilde{\ell}(\tilde{\ell} + D - 2) + \beta\alpha_2^2 \quad \text{and} \quad \Lambda'_{D-2} = \Lambda_{D-2} + \beta\alpha_2^2\,. \tag{50}$$

Equation (49) has been recently solved in $2D$ and $3D$ by the NU method in Refs. 22, 25, 26 and 30. However, the aim in this subsection is to solve it in D-dimensions. Upon letting $D = 3$, we can readily restore $3D$ solution given in Ref. 30. By comparing Eqs. (49) and (24), the corresponding polynomials are obtained

$$\tilde{\tau}(s) = -(D-1)s\,, \quad \sigma(s) = 1 - s^2\,, \quad \tilde{\sigma}(s) = -\nu' s^2 + \nu' - \Lambda'_{D-2}\,. \tag{51}$$

Inserting the above expressions into Eq. (32) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(D-3)}{2}s \pm \sqrt{\left[\left(\frac{D-3}{2}\right)^2 + \nu' - k\right]s^2 + k - \nu' + \Lambda'_{D-2}}\,. \tag{52}$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} \left(\dfrac{D-3}{2} + m'\right)s & \text{for } k_1 = \nu' - \Lambda'_{D-2}\,, \\ \left(\dfrac{D-3}{2} - m'\right)s & \text{for } k_1 = \nu' - \Lambda'_{D-2}\,, \\ \dfrac{(D-3)}{2}s + m' & \text{for } k_2 = \nu' + \left(\dfrac{D-3}{2}\right)^2\,, \\ \dfrac{(D-3)}{2}s - m' & \text{for } k_2 = \nu' + \left(\dfrac{D-3}{2}\right)^2\,, \end{cases} \tag{53}$$

where $m' = \sqrt{(\tilde{\ell}_{D-2} + D - 3/2)^2 + \beta\alpha_2^2}$. Imposing the condition $\tau'(s) < 0$ for Eq. (28), one selects the following physically valid solutions with $\tau' = \tau'(\tilde{\ell}_{D-2})$; that is, a function of the angular momentum,[c]

$$k_1 = \nu' - \Lambda'_{D-2} \quad \text{and} \quad \pi(s) = \left(\frac{D-3}{2} - m'\right)s\,, \tag{54}$$

which yields from Eq. (28) that

$$\tau(s) = -2(1 + m')s\,. \tag{55}$$

[c]The physical significance of this choice of parameters is that the eigenvalue and the eigenfunction equations can be directly reduced to the $3D$ form in Ref. 30.

Making use from Eqs. (29) and (33), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_{n_{D-1}} = 2n_{D-1}(1+m') + n_{D-1}(n_{D-1}-1)\,, \tag{56}$$

$$\lambda = \nu' - \Lambda'_{D-2} - m' + \frac{D-3}{2}\,. \tag{57}$$

Comparing Eqs. (56) and (57) and making use of the definition $\nu' = \ell'(\ell'+D-2)$, we obtain an expression for the modified angular momentum as

$$\tilde{\ell} = \tilde{\ell}_{D-1} = -\frac{(D-2)}{2} + \sqrt{\left(n_{D-1} + \frac{1}{2} + \sqrt{\left(\tilde{\ell}_{D-2} + \frac{D-3}{2}\right)^2 + \beta\alpha_2^2}\right)^2 - \alpha_2^2\beta}\,. \tag{58}$$

Equation (58) can be easily reduced to the well-known definition

$$\ell' = -\frac{1}{2} + \sqrt{\left(\tilde{\ell} + \frac{D-2}{2}\right)^2 + \beta\alpha_2^2} = n_{D-1} + m'\,, \tag{59}$$

where $n_{D-1} = n$, $\tilde{\ell}_{D-2} = \tilde{\ell}_1 = m$, $m' = \sqrt{m^2 + \alpha_2^2\beta}$ in 3D space.[30] Using Eqs. (25)–(27) and (30)–(31), we obtain

$$\phi(s) = (1-s^2)^{(2m'+3-D)/4}\,, \qquad \rho(s) = (1-s^2)^{m'}\,. \tag{60}$$

Furthermore, we substitute the wight function $\rho(s)$ given in Eq. (60) into the Rodrigues relation (30) and obtain one of the wavefunctions in the form

$$y_{n_{D-1}}(s) = B_{n_{D-1}}(1-s^2)^{-m'}\frac{d^{n_{D-1}}}{ds^{n_{D-1}}}(1-s^2)^{n_{D-1}+m'}\,, \tag{61}$$

where $B_{n_{D-1}}$ is the normaliation factor. Finally the angular wavefunction is

$$H_{n_{D-1}}(\theta_{D-1}) = N_{n_{D-1}}(\sin\theta_{D-1})^{m'-(D-3)/2}P_{n_{D-1}}^{(m',m')}(\cos\theta_{D-1})\,, \tag{62}$$

where the normalization factor

$$N_{n_{D-1}} = \sqrt{\frac{(2n_{D-1}+2m'+1)n_{D-1}!}{2\Gamma(n_{D-1}+2m')}}\,, \tag{63}$$

where m' is defined after Eq. (53).

4.2. *The eigenvalues and eigenfunctions of the radial equation*

We seek to present the exact bound state solutions, i.e., the energy spectra and radial wave function $R_{\tilde{\ell}}(r)$ of the Klein–Gordon equation in Eq. (22), by simply writting it in the following simple form:[22,25,26,41,42]

$$\frac{d^2g(r)}{dr^2} - \left[\frac{(2\tilde{\ell}+D-2)^2-1}{4r^2} + \alpha_2^2\left(Ar^2 + \frac{B}{r^2} + C\right) + \alpha_1^2\alpha_2^2\right]g(r) = 0\ ,$$

$$2\tilde{\ell} + D - 2 = \sqrt{(2n_{D-1}+1+\sqrt{(2\tilde{\ell}_{D-2}+D-3)^2+4\beta\alpha_2^2})^2 - 4\alpha_2^2\beta}\,. \tag{64}$$

For bound states, we require that the wave function $g(r)$ must satisfy the boundary condition that $g(r)$ becomes zero as $r \to \infty$, and $g(r)$ is finite at $r = 0$. Furthermore, in the application of the following transformation, $s = r^2$, and making some algebraic manipulations, we may rewrite Eq. (64) in the standard form,

$$\frac{d^2g(s)}{ds^2} + \frac{1}{2s}\frac{dg(s)}{ds} + \frac{1}{(2s)^2}[-\alpha^2 s^2 - \varepsilon^2 s - \gamma^2]g(s) = 0\,, \tag{65}$$

with the following definitions

$$\varepsilon^2 = \alpha_2^2(\alpha_1^2 + C), 4\gamma^2 + 1 = (2\tilde{\ell} + D - 2)^2 + 4B\alpha_2^2\,, \quad \alpha^2 = A\alpha_2^2\,. \tag{66}$$

Comparing Eq. (65) with Eq. (24), gives the following expressions:

$$\tilde{\tau}(s) = 1\,, \qquad \sigma(s) = 2s\,, \qquad \tilde{\sigma}(r) = -\alpha^2 s^2 - \varepsilon^2 s - \gamma^2\,. \tag{67}$$

Substituting the above expressions into Eq. (32) gives

$$\pi(s) = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\alpha^2 s^2 + 4(\varepsilon^2 + 2k)s + 4\gamma^2 + 1}\,. \tag{68}$$

Therefore, we can determine the constant k by using the condition that the discriminant of the square root is zero, that is

$$k = -\frac{\varepsilon^2}{2} \pm \frac{\alpha}{2}\sqrt{4\gamma^2 + 1}\,. \tag{69}$$

In view of that, we arrive at the following four possible functions of $\pi(s)$:

$$\pi(s) = \begin{cases} \dfrac{1}{2} + \left[\alpha s + \dfrac{1}{2}\sqrt{4\gamma^2 + 1}\right] & \text{for } k_1 = -\dfrac{\varepsilon^2}{2} + \dfrac{\alpha}{2}\sqrt{4\gamma^2 + 1}\,, \\ \dfrac{1}{2} - \left[\alpha s + \dfrac{1}{2}\sqrt{4\gamma^2 + 1}\right] & \text{for } k_1 = -\dfrac{\varepsilon^2}{2} + \dfrac{\alpha}{2}\sqrt{4\gamma^2 + 1}\,, \\ \dfrac{1}{2} + \left[\alpha s - \dfrac{1}{2}\sqrt{4\gamma^2 + 1}\right] & \text{for } k_2 = -\dfrac{\varepsilon^2}{2} - \dfrac{\alpha}{2}\sqrt{4\gamma^2 + 1}\,, \\ \dfrac{1}{2} - \left[\alpha s - \dfrac{1}{2}\sqrt{4\gamma^2 + 1}\right] & \text{for } k_2 = -\dfrac{\varepsilon^2}{2} - \dfrac{\alpha}{2}\sqrt{4\gamma^2 + 1}\,. \end{cases} \tag{70}$$

The correct value of $\pi(s)$ is chosen such that the function $\tau(s)$ given by Eq. (28) will have negative derivatives.[27] So we can choose the physical values to be

$$k = -\frac{\varepsilon^2}{2} - \frac{\alpha}{2}\sqrt{4\gamma^2 + 1} \quad \text{and} \quad \pi(s) = \frac{1}{2} - \left[\alpha s - \frac{1}{2}\sqrt{4\gamma^2 + 1}\right]\,, \tag{71}$$

which yield

$$\tau(s) = -2\alpha s + 2 + \sqrt{4\gamma^2 + 1}\,, \quad \tau'(s) = -2\alpha < 0\,. \tag{72}$$

Using Eqs. (29) and (33), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_N = 2\alpha N\,, \qquad N = 0, 1, 2, \ldots\,, \tag{73}$$

$$\lambda = -\frac{\varepsilon^2}{2} - \frac{\alpha}{2}(2 + \sqrt{4\gamma^2 + 1})\,, \tag{74}$$

where N is the radial quantum number. So we can obtain the relativistic energy levels of the Klein–Gordon as

$$-\sqrt{A}\left[4N+2+\sqrt{(2n_{D-1}+1+\sqrt{(2\tilde{\ell}_{D-2}+D-3)^2+4\beta(\mu+E_R)})^2+4(\mu+E_R)(B-\beta)}\right]$$
$$=\sqrt{\mu+E_R}(\mu-E_R+C)\,, \tag{75}$$

and hence for the pseudoharmonic plus the ring-shaped potential, it becomes

$$-\sqrt{a_0}\left[4N+2+\sqrt{(2n_{D-1}+1+\sqrt{(2\tilde{\ell}_{D-2}+D-3)^2+4\beta(\mu+E_R)})^2+4(\mu+E_R)(a_0r_0^2-\beta)}\right]$$
$$=r_0\sqrt{\mu+E_R}(\mu-E_R-2a_0)\,. \tag{76}$$

The energy E_R is defined implicitly by Eq. (76) which is rather a transcendental equation which can be easily solved numerically and its result compared with the other cases by giving some values to parameters and radial and angular quantum numbers N, $\tilde{\ell}_{D-1}$ and n_{D-1}. From the above relativistic Klein–Gordon energy eigenvalues, it is not difficult to conclude that the first energy solution is valid for the particle energy and the second one corresponds to the anti-particle energy. Further, in the non-relativistic limit, where $\mu+E_R\to 2\mu$, $\mu-E_R)\to -E_{NR}$ and $\tilde{\ell}_{D-1}\to\ell_{D-1}$, the corresponding energy eigenvalues of the Schrödinger equation with the ring-shaped pseudoharmonic potential become

$$E_{NR}=-2a_0+\sqrt{\frac{a_0}{2\mu r_0^2}}$$
$$\times\left[4N+2+\sqrt{(2n_{D-1}+1+\sqrt{(2\ell_{D-2}+D-3)^2+8\mu\beta})^2+8\mu(a_0r_0^2-\beta)}\right]\,, \tag{77}$$

which is consistent with Eq. (70) in Ref. 25. We point out that, under the above transformations, the relativistic solution given in Eq. (76) can be easily reduced into the non-relativistic limit given by Eq. (77).

For completeness, we find that it is necessary to consider the solution for the harmonic oscillator potential, $V(r)=k^2r^2/2$.[33] Therefore, applying the parameters transformation for this potential as: $A=k^2/2$, and $B=C=\beta=0$, the angular dependent potential in (4) turns into the harmonic oscillator with Klein–Gordon solution for the energy spectra as

$$(\mu+E_R)(\mu-E_R)^2=\frac{k^2}{2}[4N+2\tilde{\ell}+D]^2\,,\qquad N,\tilde{\ell}=0,1,2,\ldots\,. \tag{78}$$

On the other hand, in the non-relativistic limit, applying the following appropriate transformation: $\mu+E_R\to 2\mu$, $\mu-E_R\to -E_{NR}$, $\tilde{\ell}\to\ell$ to Eq. (78) yields

$$E_{NR}=\frac{k}{\sqrt{\mu}}\left[2N+\ell+\frac{D}{2}\right]\,,\qquad N,\ell=0,1,2,\ldots\,. \tag{79}$$

which is consistent with Refs. 25, 33 and 34. In addition, from Eq. (76), we obtain the solution for the pseudoharmonic potential ($\beta = 0$ case) in the non-relativistic $3D$ Schrödinger equation as

$$E_{NR} = -2a_0 + \sqrt{\frac{2a_0}{\mu r_0^2}} \left[2N + 1 + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + 2\mu a_0 r_0^2} \right] , \tag{80}$$

which is consistent with Ref. 34. Finally, the energy eigenvalues for the non-relativistic $3D$ Schrödinger equation with the ring-shaped pseudoharmonic potential ($\beta \neq 0$ case) is

$$E_{NR} = -2a_0 + \sqrt{\frac{2a_0}{\mu r_0^2}} \left[2N + 1 + \frac{1}{2} + \sqrt{(n + \sqrt{m^2 + 2\mu\beta})^2 + 2\mu(a_0 r_0^2 - \beta)} \right] , \tag{81}$$

where n and m are two angular quantum numbers coming from the solution of the angular wave equations.[25]

Furthermore, inserting the values of $\sigma(s)$, $\pi(s)$ and $\tau(s)$ in Eqs. (37), (40) and (41) into (27) and (31), we obtain the wavefunctions

$$\phi(s) = s^{(\zeta+1)/4} e^{-\alpha s/2} , \tag{82}$$

$$\rho(s) = s^{\zeta/2} e^{-\alpha s} , \tag{83}$$

where

$$\zeta = \sqrt{(D + 2\tilde{\ell} - 2)^2 + 4a_0 r_0^2 (\mu + E_R)} , \qquad \alpha = \frac{\sqrt{a_0(\mu + E_R)}}{r_0} . \tag{84}$$

Hence, from Eq. (30), we obtain

$$y_{N\tilde{\ell}}(s) = B_{N\tilde{\ell}} e^{\alpha s} s^{-\zeta/2} \frac{d^N}{ds^N} [e^{-\alpha s} s^{\zeta/2}] = B_{N\tilde{\ell}} L_N^{(\Xi+1/2)}(\alpha s) , \tag{85}$$

where $2\Xi + 1 = \zeta$. On the other hand, the wave function $g(s)$ can be expressed in terms of the generalized Laguerre polynomials as

$$g(s) = C_{N\tilde{\ell}} s^{(\Xi+1)/2} e^{-\alpha s/2} L_N^{(\Xi+1/2)}(\alpha s) . \tag{86}$$

Finally, the radial wave functions of the Klein–Gordon equation are obtained from Eqs. (9) and (86) as

$$R_{\tilde{\ell}}(r) = C_{N\tilde{\ell}} r^{\Xi+1-(D-1)/2} \exp(-\alpha r^2/2) L_N^{(\Xi+1/2)}(\alpha r^2) , \tag{87}$$

where[46,47]

$$C_{N\tilde{\ell}} = \sqrt{\frac{2\alpha^{\Xi+3/2} N!}{\Gamma(\Xi + N + 3/2)}} , \tag{88}$$

and we can finally obtain the re-normalized total wavefunctions

$$\begin{aligned}\psi^{(\ell)}_{\ell_1\cdots\ell_{D-2}}(\mathbf{x}) = {} & \frac{1}{2^{m'}(\tilde{n}+m')!}\sqrt{\frac{\alpha^{\Xi+3/2}N!}{\pi\Gamma(\Xi+N+3/2)}} \\ & \times r^{\Xi-(D-3)/2}\exp(-\alpha r^2/2)L_N^{(\Xi+1/2)}(\alpha r^2) \\ & \times \exp(\pm i\tilde{\ell}_1\theta_1)\prod_{j=2}^{D-2}\sqrt{\frac{(2\tilde{\ell}_j+j-1)n_j!}{2\Gamma(\tilde{\ell}_j+\tilde{\ell}_{j-1}+j-2)}} \\ & \times (\sin\theta_j)^{\tilde{\ell}_j-n_j}P_{n_j}^{(\tilde{\ell}_j-n_j+(j-2)/2,\tilde{\ell}_j-n_j+(j-2)/2)}(\cos\theta_j) \\ & \times \sqrt{\frac{(2n_{D-1}+2m'+1)n_{D-1}!}{2\Gamma(n_{D-1}+2m')}}(\sin\theta_{D-1})^{m'-(D-3)/2} \\ & \times P_{n_{D-1}}^{(m',m')}(\cos\theta_{D-1})\,, \end{aligned} \tag{89}$$

where $\Xi = (\zeta-1)/2$.

5. Conclusions

We have calculated the exact bound state energy eigenvalues and the corresponding wave functions of the relativistic spin-0 particle in the D-dimensional Klein–Gordon equation with equal scalar and vector ring-shaped pseudoharmonic potential using the NU method. The analytical expressions for the total energy levels and wave functions of this system can be reduced to their well-known 3D-Schrödinger equation. Furthermore, the angular dependent potentials treated in Ref. 20 can be introduced as perturbation to the pseudoharmonic potential by adjusting the strength of the coupling constant β in terms of a_0, which is the coupling constant of the pseudoharmonic potential. The relativistic energy E_R defined implicitly by Eq. (75) is rather a transcendental equation and it has many solutions for any arbitrarily given values of quantum numbers N, $\tilde{\ell}_{D-2}$ and n_{D-1}. We point out that the radial and angular wave functions of the Klein–Gordon equation are found in terms of Laguerre and Jacobi polynomials, respectively. The method presented in this paper is general and worth extending to the solution of other interaction problems. This method is simple and useful in solving other complicated systems analytically without giving a restiction condition on the solution of some quantum systems as the case in the other models. We have also seen that for the non-relativistic model, the exact energy spectra can be obtained by either solving the Schrödinger equation in Eq. (11) (cf., Eq. (48) in Ref. 25) or applying appropriate transformation to the relativistic solution. We should emphasize that the exact bound state energy spectra obtained in this work might have some interesting applications in different branches like atomic and molecular physics and quantum chemistry. This solution describes the molecular structures and interactions in diatomic molecules.[48]

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