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Error Analysis For The Finite Element Approximation of Conductive-Radiative Model

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Abstract

In this article we focus our attention on the finite element error analysis for a problem involving both conductive and radiative heat transfer. We sketch the main steps of the analysis by stating the required a priori estimates and the final estimates. The proof for the estimate of the error due to approximation of the geometry is also presented. We prove an abstract estimate for the discretization error in a polygonal domain and combine it to the geometric estimate to yield the final error estimate. A concrete inverse monotone numerical method using view factors is analyzed using the abstract estimates.

Keywords: Heat radiation, non-local problems, finite element method.

Introduction

Heat radiation is a significant factor in heat transfer in many situations, typically always when a hot surface is in contact with a transparent medium with relatively law heat conductivity. The physical principles of heat radiation are well understood and a chapter on radiative heat transfer can be found in most books on heat transfer. Also the numerical treatment of the problem is a popular subject in the engineering literature, see for example [1], [4], [11] and [12] and references therein.
On the other hand, mathematical papers dealing with questions related to heat radiation or its numerical treatment are relatively hard to find. Moreover, the papers tend to focus on the simplest possible case of a convex body radiating heat to infinity, see for example [3], [6] and [8, 9, 10].
In this article we focus our attention on the numerical approximation of the model for heat transfer by conduction and radiation. The main characteristic feature of such a model is the non-locality arising from radiation exchange between different parts of the system. The structure of the paper is as follows: In section 2, we present the model describing the heat transfer by conduction and radiation. In section 3, we sketch the main steps of the analysis by stating the required a priori estimates and the final estimates. Then in section 4 we present the proof for the estimate of the error due to approximation of the geometry. In section 5 we prove an abstract estimates for the descritization error in a polygonal domain and combine it to the geometric estimate to yield the final abstract error estimate. Then in section 6 we analyze a concrete inverse monotone numerical method using view factors method using the abstract estimates. Throughout this paper, we shall use the standard Lebesgue and Sobolev spaces with the following notations: By $L^p(\Omega)$ we denote the space of measurable functions ψ for which

$\|\psi\|_{L^p} = \left(\int_{\Omega} |\psi|^p \right)^{1/p}$ is finite. By $W^{m,p}(\Omega)$ we denote the space of $L^p(\Omega)$ functions whose generalized derivatives up to order m are in $L^p(\Omega)$. The space $W^{m,p}(\Omega)$ is equipped with norm $\|\psi\|_{m,p} = \sum_{k=0}^{m} \|D^k \psi\|_{L^p}$. Notation $H^m(\Omega)$ with norm $\|\psi\|_m$ is used for $W^{m,p}(\Omega)$. In the Hilbertian case also the Sobolev spaces with real-valued indices will be needed. As a rule the domains are not indicated explicitly in the norms. By the space $H^1(\Omega;\Sigma)$ we refer to the space $\{\psi \in H^1(\Omega), \psi|_{\Sigma} = 0\}$.

The physical model

We consider the model problem

$$\int_{\Omega} \nabla T \nabla \psi + \int_{\Gamma} \sigma (I - K) T^4 \psi = \langle g, \psi \rangle \quad ; \quad \forall \psi \in W \tag{2.1}$$

with $T - T_0 \in W = \{H^1(\Omega) \mathrm{I}\ L^5(\Gamma)\} \cdot$ σ is the Stefan-Boltzmann constant which has the value $5.6696 \times 10^{-8} \left(\mathrm{W}/m^2 K^4 \right)$ and K is the integral operator modeling the non-local heat transfer due to radiations between different part of the surface Γ. We consider here the case of black diffuse surfaces without shadowing zones. In this case this integral operator K can be written in three-dimensional geometry as [8]

$$Kq_0(x) = \int_{\Gamma} \frac{n_x \cdot (y - x)\, n_y (y - x)}{\pi |y - x|^4} q(y) dy . \tag{2.2}$$

The properties of this integral operator have been thoroughly investigated in [10] and [13]. For the sake of clarity, we list the following properties [10]:

- (i) $K \geq 0$
- (ii) K is compact from $L^p(\Gamma)$ into itself.
- (iii) To $\|K\|_{L^p(\Gamma)} \leq 1$, $1 \leq p \leq \infty$, strictly if Γ is not a closed surface.
- (iv) $\int_{\Gamma} T^5 - (KT^4) \geq 0$; $\forall T \geq 0$, strictly if Γ is not a closed surface and $T \neq 0$.

These properties imply that the problem (2.1) is coercive in W and pseudo-monotone. Due to the non-monotonicity of this problem the standard technique for approximating strongly monotone problems can not be applied. Since problem (2.1) has a maximum principle, we are led to study approximation methods that lead to discrete problems satisfying a corresponding discrete maximum principle. In the case of pure heat conduction this problem is well studied and it is well known that discrete maximum principle can be obtained using simplical elements in polygonal domains if the finite mesh satisfies some geometrical constraints [2] and [5]. Due to the non-convexity of the boundary sufficient regularity can not be expected from the solution of polyhedral geometries. Hence, we have to address also the problem of approximating the domain.

This question is quite complicated by the presence of the integral operator K acting on the boundary. That is why we shall separate the analysis of the 'geometrical' error from the analysis of the finite element discretization. This means that we study the continuous dependence on geometric data for the continuous problem (2.1). The techniques needed for estimating the error in the geometry are based on shape optimization. The non-monotonicity creates also some additional problems as the linearizations of (2.1) are not necessarily coercive. Thus the solvability of the pertubation equations must be based on the maximum principle.

Main results

We assume that Ω is a smooth domain. In this case Ω can not be exactly discretized using simplical elements that we want to use. Thus we have to consider the error due to approximating Ω by Ω_h. For that purpose we define an auxiliary problem in Ω_h that will later be discretized. We assume that the distance of $\partial\Omega$ and $\partial\Omega_h$ is smaller than the local curvature of $\partial\Omega$ so that $\partial\Omega_h$ can be represented as

$$\partial\Omega_h = \left\{ x + u(x)\cdot n(x) \middle| \, x \in \partial\Omega \right\} \tag{3.1}$$

where n denotes the outer unit normal of $\partial\Omega$. Using [7] we can extended u as $W^{1,\infty}$ function $\tilde{u}$ that is piecewise C^2 to the neighborhood of $\Omega \cup \Omega_h$. We denote by U the scaled extension, $U = \tilde{u}|h$, so that $hU|_{\partial\Omega} = u$. Using the vector field U we can define a family of deformations of Ω:

$$\Omega_t = M_t(\Omega),\ M_t(x) = x + tU(x). \tag{3.2}$$

In fact $\Omega_h = \Omega_t|_{t=h}$. For the deformation velocity U we have the estimates

$$\| U \|_{L^\infty} \le C\,h\ ,\ \ \|U\|_{1,\infty} \le C\,. \tag{3.3}$$

To this end, the auxiliary problem in Ω_t is

$$\int_{\Omega_t} \nabla T_t \nabla \psi + \int_{\Gamma_t} \sigma(I-K)\,(T_t)^4 \psi = \langle g_t, \psi \rangle \qquad \forall\, \psi \in W_t = W(\Omega_t) \tag{3.4}$$

with $T_t|_{\Sigma_t} = \tilde{T}_0|_{\Sigma_t}$ and $\langle g_t, \psi \rangle = \int_{\Omega_t} \tilde{g}\psi$, where $\sim$ denotes appropriate extensions to D. Since T and T_t are defined as different domains one can not compare them directly. Instead we estimate $\tilde{T} - T_t$ where $\tilde{T}$ is a smooth extension of T. For our purposes it is, however, sufficient and simpler to estimate that error between T and $T_t \circ M_t$ as for smooth T we have

$$\left\| \tilde{T} - T_0 M_t^{-1} \right\|_1 \le Ct \| T \|_2 .$$

Theorem 1 If $T \in W \cap L^\infty$, $T_t \in W_t \cap L^\infty$, $g \in L^2$, $\left\| T - \tilde{T}_0 \circ M_t \right\|_1 \le C\,t$ and $\tilde{g}$ is the L^2 extension of g, then for the error between T and T_t it holds

(3.5)

To this end, let us now consider the finite element discretization of the auxiliary problem in Ω_h. We triangulate Ω_h using simplical elements with characteristic grid size h. Furthermore, we choose the triangulation to be compatible with the discrete maximum principle [2] and [5]. Likewise, we discretize the radiative part so that the discrete maximum principle remains valid; for more details see section 5. For the discretization error we have the following priori estimate.

Theorem 2 Assume that the finite element grid consists of simplices with a cute angles only and that the discrete version $I-K$ is singular M-matrix, then for the error between the auxiliary solution $\hat{T}=T_t\big|_{t=h}$ and its finite element approximation T_h it holds

$$\left\|\hat{T}-T_h\right\|_1 \le C \inf_{\psi_h \in W_h}\left(\left\|\hat{T}-T_h\right\| + \sup_{\psi_h \in W_h} \frac{\left|\int_{\Gamma_h}(I-K)\varphi_h^4\psi_h dx_h - \int_{\Gamma_h}(I-K_h)\varphi_h^4\psi_h dx_h\right|}{\left\|\psi_h\right\|_1}\right), \tag{3.6}$$

where dx_h and K_h refer to numerical integration and to numerically evaluated integral operator respectively.

In order to obtain the final estimates we have to use the estimate of (3.5) into (3.6) to improve the interpolation estimates using the higher regularity of the original solution. Thus the final estimates are given in the following theorem.

Theorem 3 Assume that the hypotheses of both theorems 1 and 2 are satisfied. Then for the error between the solution of (2.1) and its finite element approximation it holds

$$\left\| T - T_h \circ M_h \right\|_1 \le Ch\left\| T \right\|_2, \tag{3.7}$$

$$\left\| \widetilde{T} - T_h \right\|_1 \le Ch\left\| T \right\|_2. \tag{3.8}$$

The geometric error estimation

We consider

$$a_t(T_t,\psi) = \int_{\Omega_t} \nabla T_t \psi \ , \tag{4.1}$$

$$b_t(T_t,\psi) = \int_{\Gamma_t} (I-K_t)T_t^4 \psi \tag{4.2}$$

where $\psi \in W(\Omega_t)$ and $T_t - \widetilde{T}_0 \in W(\Omega_t)$. In order to estimate the difference between $T = T_0$ and T_t we have to bring them to the same domain of definition. In fact it is convenient to transport T_t to Ω. We denote by a^t and b^t the transported versions of a_t and b_t, and by $T^t = T_t \circ M_t^{-1}$ the transported solution, then since

$$a^t(T^t,\psi)+b^t(T^t,\psi)=\left\langle g^t,\psi\right\rangle \tag{4.3}$$

and

$$a(T,\varphi)+b(T,\varphi)=\langle g,\psi\rangle \tag{4.4}$$

we have

$$\begin{aligned} a(T^t-T,\psi)+\widetilde{b}(T^t-T,\psi)=\left\langle g^t-\mathrm{g},\psi\right\rangle+a(T^t,\psi)-a^t(T^t,\psi)\\ +b(T^t,\psi)-b^t(T^t,\psi)\qquad \forall\,\psi\in W,\end{aligned} \tag{4.5}$$

where $\widetilde{b}(q,\psi)=\int_\Gamma (I-K)4\widetilde{T}^3 q\psi$, $\widetilde{T}$ is such that $T^4-(T^t)^4=4\widetilde{T}^3(T-T^t)$ and T^t and $T^t-T-(\widetilde{T}_0 \circ M_t^{-1}-T_0)\in H^1(\Omega,\Sigma)$. Then, to estimate T^t-T we have to bound the right hand side of (4.5) and to show that $a+\widetilde{b}$ is continuously invertible in suitable norms. As a is strongly elliptic in $H^1(\Omega,\Sigma)$ one can find δ so that the problem

$$a(q,\psi)+\widetilde{b}(q,\psi)+\delta(q,\psi)_{L^2}=\langle \mathrm{g},\psi\rangle \qquad \forall\,\psi\in H^1(\Omega;\Sigma), \tag{4.6}$$

where $q\in H^1(\Omega,\Sigma)$ admits a unique solution.

Lemma 1 Assume that $g\in\left(H^1(\Omega;\Sigma)\right)'$, $g\le 0$ and $q_0\le 0$, $q_0\in H^1(\Omega)$ are given. Let further $\widetilde{T}\in L^\infty$, $\widetilde{T}>0$. Then for the solution q of the problem

$$a(q,\psi)+\widetilde{b}(\psi,q)=\langle \mathrm{g},\psi\rangle \qquad \forall\,\psi\in H^1(\Omega;\Sigma), \tag{4.7}$$

with the condition $q-q_0\in H^1(\Omega;\Sigma)$, it holds $q\le 0$ in Ω.
Hence with the Fredholm alternative we obtain that the problem

$$a(q,\psi)+\widetilde{b}(q,\psi)=\langle \mathrm{g},\psi\rangle \qquad \forall\,\psi\in H^1(\Omega;\Sigma), \tag{4.8}$$

with $q-q_0\in H^1(\Omega;\Sigma)$ has a unique solution q in $H^1(\Omega)$ for any $g\in\left(H^1(\Omega;\Sigma)\right)'$ and any $q_0\in H^1(\Omega)$. Thus

$$\|q\|_{H^1}\le C\left(\|q_0\|_{H^1(\Omega)}+\|g\|_{(H^1(\Omega;\Sigma))'}\right). \tag{4.9}$$

To estimate the right hand side of (4.5) we have to define a^t and b^t explicitly. To do that we need some basic results concerning the deformation of M_t.

It holds for $g\in L^1(\Omega_t)$ and $f\in L^1(\Gamma_t)$ that

$$\int_{\Omega_t} g\,dx=\int_\Omega g\circ M_t\left|DM_t\right|dx, \tag{4.10}$$

$$\int_{\Gamma_t} f\,ds=\int_\Gamma f\circ M_t\left|D\Gamma_t\right|ds \tag{4.11}$$

where $\left|D\Gamma_t\right|=\left|DM_t\right|\left\|DM_t^{-\mathsf{T}}n_\Gamma\right\|$. For the transported gradient we have

$$\nabla g\circ M_t=DM_t^{-1}\nabla(g\circ M_t) \tag{4.12}$$

where $g\in W^{1,1}(\Omega)$. Finally, if n_t is the unit normal to Γ_t then

$$n_t\circ M_t=\left\|DM_t^{-1}n\right\|^{-1}DM_t^{-1}n. \tag{4.13}$$

To this end, we can now study the terms on the right hand side of (4.5). First for the difference between a and a^t we have

Lemma 2 For any $\psi \in H^1(\Omega;\Sigma)$ we have

$$\left|(aT^t,\psi) - a^t(T^t,\psi)\right| \le C(t+o(t))\,\|\mathrm{DU}\|_{L^\infty}\left\|\nabla T^t\right\|_{L^2}\|\nabla\psi\|_{L^2}. \tag{4.14}$$

Proof: We have

$$a^t(T^t,\psi) = \int_\Omega DM_t^{-\mathsf{T}}\nabla T^t DM_t^{-\mathsf{T}}\nabla\psi\,|DM_t|\,dx \tag{4.15}$$

Hence as $DM_t = I + t\,DU$ we have that

$$\left\|DM_t^{-1} - I\right\|_{L^\infty} = (t+o(t))\,\|DU\|_{L^\infty}$$

and

$$\left|\,|DM_t| - 1 - tT_n DU\right| \le \mathrm{o(t)}\|DU\|_{L^\infty}.$$

By straight forward substitution we observe that

$$(aT^t,\psi) - a^t(T^t,\psi) = t\int_\Omega DU\nabla T^t\nabla\psi + \nabla T^t DU\nabla\psi + \nabla T^t\nabla\psi T_r DU + o(t). \tag{4.16}$$

Hence the conclusion follows easily.
Similarly, for the radiation term we have the following estimates.

Lemma 3 For all $\psi \in W$ we have

$$(bT^t,\psi) - b^t(T^t,\psi) \le C(t+o(t))\,\|\mathrm{DU}\|_{L^\infty}\left\|T^t\right\|^4_{L^5(\Gamma)}\|\psi\|_{L^5(\Gamma)}. \tag{4.17}$$

Furthermore, if $T^t \in L^{16/3}(\Gamma)$, then for all $\psi \in H^1(\Omega;\Sigma)$ we have

$$b(T^t,\psi) - a^t(T^t,\psi) \le C(t+o(t))\,\|\mathrm{DU}\|_{L^\infty}\left\|T^t\right\|^4_{L^{16/3}(\Gamma)}\|\psi\|_1. \tag{4.18}$$

Finally, we have to estimate the difference arising from the data term.

Lemma 4 For all $\psi \in H^1(\Omega;\Sigma)$ we have

$$\left|\left\langle g^t - g,\psi\right\rangle\right| \le Ct\|\mathrm{U}\|_{L^\infty}\|g\|_{L^2}\|\psi\|_1. \tag{4.19}$$

Proof: Let us denote by $\tilde{\psi}$ an H^1-extension of ψ to D.
Then as

$$\left\langle g^t,\psi\right\rangle = \int_\Omega \tilde{g}\circ M_t\,\psi\,|DM_t|\,dx \tag{4.20}$$

we can write

$$\left|\left\langle g^t,\psi\right\rangle\right| \le \int_D \left|\tilde{g}\circ M_t\,\tilde{\psi}\right|\left|DM_t\right| - \left|\tilde{g}\tilde{\psi}\right|. \tag{4.21}$$

By taking D to be large enough we can choose the extension of u in such a way that $U = 0$ on ∂D.
Hence as M_t maps D into itself and

$$\int_D \tilde{g}\circ M_t\,|DM_t|\tilde{\psi} = \int \tilde{g}\tilde{\psi}\circ M_t^{-1}. \tag{4.22}$$

Thus, as

$$\int_D \left|\tilde{g}(\tilde{\psi}\circ M_t^{-1} - \tilde{\psi}\right| \le (t+o(t))\,\|g\|_{L^2}\|\mathrm{U}\|_{L^\infty(D)}\|\tilde{\psi}\|_{H^1(D)}, \tag{4.23}$$

the conclusion follows.

Abstract estimate

In this section we shall consider an a priori error estimate for the difference between the finite element solution and the solution of the auxiliary problem in Ω_h. We denote by W_h a finite element subspace of $W(\Omega_h)$. We consider the following abstract discrete problem

$$a(T_h,\psi_h)-b_h(T_h,\psi_h)=\langle g,\psi_h\rangle \qquad \forall\psi_h\in W_h \tag{5.1}$$

with $T_h-T_{oh}\in W_h$ and b_h is a discretization of b.

We denote by $\hat{T}=T_t\big|_{t=h}$ the solution of the auxiliary problem (3.4). To estimate the error between $\hat{T}$ and T_h we write first

$$\left\|\hat{T}-T_h\right\|_1 \le \left\|\hat{T}-\varphi_h\right\|_1+\left\|T_h-\varphi_h\right\|_1, \tag{5.2}$$

where φ_h is some approximation of $\hat{T}$. Next we write an equation for $T_h-\varphi_h$,

$$a(T_h-\varphi_h,\psi_h)+\widetilde{b}_h(T_h-\varphi_h,\psi_h)=a(\hat{T}-\varphi_h,\psi_h)+b(\hat{T},\psi_h)-b_h(\varphi_h,\psi_h). \tag{5.3}$$

Where $\widetilde{b}_h=\widetilde{c}_h-\widetilde{d}_h$ with $\widetilde{c}_h(T_h,\psi_h)=\int_{\Gamma_h}4\widetilde{T}_h^3\psi_h dx_h$ and $\widetilde{T}_h$ is such that

$T_h^4-\varphi_h^4=4\widetilde{T}_h^3(T_h-\varphi_h)$. Likewise, $\widetilde{d}_h(T_h,\psi_h)=\int_{\Gamma_h}K4\widetilde{T}_h^3T_h\psi_h dx_h$. To derive a bound for $T_h-\varphi_h$ we estimate the right hand side of (5.3) and bound the solution operator of $a+\widetilde{b}_h$. We assume that both discrete problems (5.1) and (5.3) are solvable.

Lemma 5 Assume that both the nonlinear and linearized discrete problems are solvable. Furthermore, assume that T_h are uniformly bounded in L^∞ and that the inverse of $a+\widetilde{b}_h$ is bounded independently of h as a mapping from $\left(H^1(\Omega;\Sigma)\right)'$ to $H^1(\Omega;\Sigma)$. Then, for the error between $\hat{T}$ and T_h it holds

$$\left\|\hat{T}-T_h\right\|_1 \le C\inf_{\varphi_h\in W_h}\left(\left\|\hat{T}-\varphi_h\right\|_1+\sup_{\psi_h\in W_h}\frac{\left|b(\varphi_h,\psi_h)-b_n(\varphi_h,\psi_h)\right|}{\left\|\psi_h\right\|_1}\right). \tag{5.4}$$

Proof: It is very clear that by (5.2) will by sufficient to derive the estimate for $\left\|\hat{T}-\varphi_h\right\|_1$ only. Furthermore, since the linearized problem has a uniform continuous inverse, we only have to bound the right hand side of (5.3). For the term involving a we use the continuity of a. For the terms involving b and b_h we add and subtract $b(\varphi_h,\psi_h)$. Thus we observe that $\left|b(\widetilde{T},\psi_h)-b(\varphi_h,\psi_h)\right|\le C\left\|\varphi_h-\hat{T}\right\|_1\left\|\psi_h\right\|_1$ as $\hat{T}$ and φ_h are bounded in L^∞. We are now ready to formulate an abstract version of theorem 3.

Lemma 6 Assume that the conditions of Lemma 5 and Theorem 1 hold. Then for the error between the solution of the original problem and its finite element approximation it holds,

$$\left\|T-T_h\,\mathrm{o}M_h\right\|_1 \le C\left(\left\|T-\hat{T}_h\,\mathrm{o}M_h\right\|_1+\inf_{\varphi_h\in W_h}\left(\left\|T\,\mathrm{o}M_h^{-1}-\varphi_h\right\|_1+\sup_{\psi_h\in W_h}\frac{\left|b(\varphi_h,\psi_h)-b_n(\varphi_h,\psi_h)\right|}{\left\|\psi_h\right\|_1}\right)\right) \tag{5.5}$$

Proof: Let us start by writing

$$\left\|T-T_h\,\mathrm{o}M_h\right\|_1 \le \left\|T-\hat{T}_h\,\mathrm{o}M_h\right\|_1+\left\|\hat{T}\,\mathrm{o}M_h-T_h\,\mathrm{o}M_h\right\|_1. \tag{5.6}$$

Then, as M_h transforms H^1 – functions to H^1, we only have to apply the priori estimate of Lemma 5. However, this is not straight forward since the auxiliary solution of $\hat{T}$ has not the regularity needed for optional interpolation estimates. To overcome this regularity problem we estimate as follows

$$\left\|\hat{T}-\varphi_h\right\|_1 \le \left\|\hat{T}-T \circ M_h^{-1}\right\|_1 + \left\|T \circ M_h^{-1}-\varphi_h\right\|_1 . \tag{5.7}$$

Hence, from Lemma 5 we get the conclusion.

Concrete scheme

Let τ_h be a regular decomposition of Ω_h into a regular tetrahedral. As the space W_h we choose the space of piecewise linear functions

$$W_h = \left\{\psi_h \in C^o(\overline{\Omega}_h) \left| \psi_h\right|_E \in P^1(E),\ \ \forall E \in \tau_h, \psi_h|_{\Sigma_h} = 0\right\}, \tag{6.1}$$

where $C^o(\overline{\Omega}_h)$ is the set of continuous functions in $\overline{\Omega}_h$ and P^1 is the space of first order polynomials.

To this end the discrete problem takes the form

$$a(T_h,\psi_h) - b_h(T_h,\psi_h) = \left\langle g, \psi_h \right\rangle \qquad \forall \psi_h \in W_h, \tag{6.2}$$

$T_h - T_{oh} \in W_h$, with T_{oh} is a some piecewise linear approximation of T_0 satisfying $\left\|T_0 - T_{oh}\right\|_1 \le Ch$.

The definition of b_h must be made with the special case. First we write

$$b(T_h,\psi_h) = \int_{\Gamma_h} \sigma T_h^4 \psi_h - \int_{\Gamma_h} \sigma \int_{\Gamma_h} G(y,x) T_h^4(x)\, dx\, \psi_h(y)\, dy . \tag{6.3}$$

We approximate b by

$$\widetilde{b}_n(T_h,\psi_h) = \int_{\Gamma_h} \sigma P_h\,(T_h^4 \psi_h) - \int_{\Gamma_h} \sigma \int_{\Gamma_h} G(y,x) \overline{P_h\,(T_h^4 \psi_h)}(x) dx \overline{\psi_h(y)} dy \tag{6.4}$$

where P_h is the interpolation operator to W_h and by $\left(\overline{\,\cdot\,}\right)$ we denote the L^2 - projection to piecewise constants. The above discretization method is a variant of the method of the so-called view factors. In practice it amounts to evaluating double surface integrals over all surface element pairs. That is, to construct the matrix M as

$$M_{ij} = \int_{\Gamma_i} \int_{\Gamma_j} G(y,x)\, dy dx . \tag{6.5}$$

Now we can see that $M_{ij} = DF$ with $D = \mathrm{Diag}\left\{ \left|\Gamma_i\right| \right\}$ and F_{ij} is the view factor between the surface elements i and j which indicates how much radiation leaving from Γ_j is intercepted by Γ_i [11].

To define the final discrete radiation term b_n we assume that in (6.4) the quadrature method has been chosen to that the numerically evaluated matrix $\widetilde{M}$ satisfies

$$\left| \left(M_{ij} - \widetilde{M}_{ij}\right) \right| \le C \max(h^5, hM_{ij}) \qquad i \ne j . \tag{6.6}$$

This can be achieved with appropriate adaptive quadrature schemes. Moreover, in order to guarantee that the discrete problem satisfies the maximum principle, we have to control the row sums of $\widetilde{M}$. We require that $\sum_j \widetilde{M}_{ij} \le \left|\Gamma_i\right|$. This can be obtained by an $O(h^3)$ modification of $\widetilde{M}_{ii}$. Note that $M_{ii} = 0\ \forall i$ for polyhedral domains. This, together with (6.6) implies that $\left\|M - \widetilde{M}\right\|_p \le C h^3$ where $\left\|\cdot\right\|_p$ denotes a matrix norm.

Now, we need to consider the approximation of T_h^4 and ψ_h by piecewise constants. Let us denote by $\overline{\psi}_h$ the L^2 – projection of ψ_h into piecewise constants on Γ_h.

Lemma 7 For all $\psi_h, \varphi_h \in H^{1/2}(\Gamma_h)$ it holds

$$\left| \left(K\psi_h, \varphi_h\right) - \left(K\overline{\psi}_h, \overline{\varphi}_h\right) \right| \le Ch \|\psi_h\|_{1/2} \|\varphi_h\|_{1/2}. \tag{6.7}$$

Proof: Next we write

$$\left(K\psi_h, \varphi_h\right) - \left(K\overline{\psi}_h, \overline{\varphi}_h\right) = \left(K\left(\psi_h - \overline{\psi}_h\right), \varphi_h\right) + \left(K\left(\varphi_h - \overline{\varphi}_h\right), \varphi_h\right) + \left(K\left(\psi_h - \overline{\psi}_h\right), \varphi_h - \overline{\psi}_h\right).$$

From approximation theory we know that

$\|\varphi_h - \overline{\varphi}_h\|_0 \le Ch^{1/2} \|\varphi_h\|_{1/2}$ and $\|\varphi_h - \overline{\varphi}_h\|_{-1/2} \le Ch \|\varphi_h\|_{1/2}$. Since K maps L^2 into L^2, we get that

$$\left(K\left(\psi_h - \overline{\psi}_h\right), \varphi_h - \overline{\psi}_h\right) \le C \|\varphi_h - \overline{\varphi}_h\|_0 \|\psi_h - \overline{\psi}_h\|_0 \le Ch \|\psi_h\|_{1/2} \|\varphi_h\|_{1/2}. \tag{6.8}$$

For C^2-domain it is known [8] that K maps L^2 into H^2. So let us write

$$\begin{aligned}\left(K\varphi_h, \psi_h - \overline{\psi}_h\right) &= \left(M_h \circ \widetilde{K}\varphi_h \circ M_h^{-1}, \psi_h - \overline{\psi}_h\right) \\ &\quad + \left(K\varphi_h - M_h \circ \widetilde{K}\varphi_h \circ M_h^{-1}, \psi_h - \overline{\psi}_h\right)\end{aligned} \tag{6.9}$$

Where $\widetilde{K}$ is the operator corresponding to Γ. Thanks to the regularity properties of $\widetilde{K}$ we can use the $H^{-1/2}$ projection estimate in the first term. In the second term we observe from Lemma 3 that $\left\|K\varphi_h - M_h \circ \widetilde{K}\varphi_h \circ M_h^{-1}\right\|_0 \le Ch \|\varphi_h\|_0$.

Lemma 8 Let $T_h \in W_h \cap L^\infty$, $\psi_h \in W_h$. Then

$$\left|b(T_h, \psi_h) - b_h(T_h, \psi_h)\right| \le Ch \|\psi\|_{1/2} \left(\left\|T_h^4\right\|_{1/2} + \left\|T_h\right\|_{L^\infty}^2 \left\|T_h\right\|_{1/2}^2 \right). \tag{6.10}$$

Proof: First we write

$$\begin{aligned} b(T_h, \psi_h) - b_h(T_h, \psi_h) &= \int_{\Gamma_h} \sigma\left(T_h^4 \psi_h - P_h(T_h^4 \psi_h)\right) - \int_{\Gamma_h} \sigma K\left(T_h^4 - P_h T_h^4\right) \psi_h \\ &\quad - \int_{\Gamma_h} \sigma\left(K P_h T_h^4 \psi_h - K \overline{P_h T_h^4} \overline{\psi}_h\right) - \vec{\psi}(M - \widetilde{M}) \vec{T}^4, \end{aligned} \tag{6.11}$$

where $\vec{\psi}$ denotes the vector corresponding to $\overline{\psi}_h$ in the canonical basis of piecewise constants on Γ_h. Now by standard arguments,

$\left| \int_{\Gamma_h} \sigma\left(T_h^4 \psi_h - P_h(T_h^4 \psi_h)\right) \right| \le Ch \left\|T_h^4\right\|_{1/2} \left\|\psi_h\right\|_{1/2}$. For the second integral on the right hand side of (6.11) it suffices to estimate $T_h^4 - P_h T_h^4$ in L^2. As

$$\left\|T_h^4 - P_h T_h^4\right\|_{L^2}^2 \le Ch^4 \sum_{E \in \mathcal{T}_h} \left|T_h^4\right|_{2, \overline{E} \cap \Gamma_h}^2 \tag{6.12}$$

and

$$\left|T_h^4\right|_{2, \overline{E} \cap \Gamma_h} \le C \left\|T_h^2 |\nabla T_h|^2\right\|_{L^2, \overline{E} \cap \Gamma_h}^2 \le C \left\|T_h\right\|_{L^\infty}^2 \left\|\nabla T_h\right\|_{L^2}^2 \tag{6.13}$$

we obtain using the inverse estimate $\|\nabla T_h\|_{L^2}^2 \le Ch^{-1} \|T_h\|_{1/2}^2$ that

$$\left\| T_h^4 - P_h T_h^4 \right\|_{L^2} \leq Ch \left\| T_h \right\|_{L^\infty}^2 \left\| T_h \right\|_{1/2}^2. \tag{6.14}$$

From Lemma 7 we get

$$\int_{\Gamma_h} \sigma \left(KP_h T_h^4 \psi_h - K \overline{P_h T^4} \overline{\psi}_h \right) \leq Ch \left\| T_h^4 \right\|_{1/2} \left\| \psi_h \right\|_{1/2}. \tag{6.15}$$

Finally as $\left\| M - \widetilde{M} \right\|_2 \leq Ch^3$ we obtain $\left| \overset{\rho}{\psi}(M - \widetilde{M}) \overset{\rho}{T}^4 \right| \leq Ch \left\| T_h^4 \right\|_0 \left\| \psi_h \right\|_0$.

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