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RESEARCH ARTICLE

Approximate l-state solution of the trigonometric Pöschl–Teller potential

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The trigonometric Pöschl–Teller (PT) potential describes diatomic molecular vibration. We have obtained the approximate solutions of the radial Schrödinger equation for the rotating trigonometric PT potential using the Nikiforov–Uvarov method. The energy eigenvalues and their corresponding eigenfunctions are calculated for arbitrary l-states in closed form. In the low-screening region, when the screening parameter $\alpha \to 0$, the potential reduces to the Kratzer potential. Numerical results are presented for several diatomic molecules.

Keywords: Schrödinger equation; trigonometric Pöschl–Teller (PT) potential; Kratzer potential; Nikiforov–Uvarov method

1. Introduction

The solution of the fundamental dynamical equations is an interesting phenomenon in many fields of physics and chemistry. The exact solutions of the Schrödinger equation (SE) for a hydrogen atom (Coulombic) and for a harmonic oscillator represent two typical examples in quantum mechanics [1–3]. The Mie-type and pseudoharmonic potentials are also two exactly solvable potentials [4, 5]. Many authors have exactly solved SE with different potentials and methods [6–16].

The trigonometric Pöschl–Teller (PT) potential was proposed by Pöschl and Teller [17] in 1933 to describe diatomic molecular vibration. Chen [18] and Zhang and Wang [19] studied the relativistic bound state solutions for the trigonometric PT potential and hyperbolic PT (second PT) potential, respectively. Liu \textit{et al.} [20] studied the trigonometric PT potential within the framework of Dirac theory. Very recently, Hamzavi and Rajabi studied the exact s-wave solution ($l = 0$) of the Schrödinger equation for the vibrational trigonometric PT potential [21]. This potential takes the form

$$V(r) = \frac{V_1}{\sin^2(ar)} + \frac{V_2}{\cos^2(ar)}, \quad V_1 > 0, V_2 > 0, \quad (1)$$

where the parameters $V_1$ and $V_2$ describe the properties of the potential well, and $\alpha$ is related to the range of this potential [20]. We find that this potential has a minimum value at $r_0 = (\pi/4\alpha) \in (0, \infty)$ for $\alpha > 0$. The second derivative, which determines the force constants at $r = r_0$, is given by

$$\frac{d^2V}{dr^2}\bigg|_{r=r_0} = \frac{8\alpha^2(V_2 + \sqrt{V_1V_2})}{\cos^2[\tan^{-1}(\sqrt{V_1/V_2})]}, \quad (2)$$

for any $\alpha$ value, and

$$V(r_0) = \frac{\sqrt{V_1V_2} + V_2}{\cos^2[\tan^{-1}(\sqrt{V_1/V_2})]}, \quad (3)$$

which means that $V(r)$ at $r = r_0$ has a relative minimum for $\alpha > 0$. When $V_1 = V_2 = V$, the minimum value is $V(r_0) = 4V$ and $d^2V/dr^2\big|_{r=r_0} = 32\alpha^2V$. Figures 1(a) and (b) show the trigonometric PT potential (1) for parameter values $V_1 = 5.0 \text{ fm}^{-1}$, $V_2 = 3.0 \text{ fm}^{-1}$, $\alpha = 0.02 \text{ fm}^{-1}$ and $\alpha = 0.30 \text{ fm}^{-1}$. Here, the potential has a minimum value at $r_0 = 0.27027\pi/\alpha$. The curve is nodeless in $ar \in (0, \pi/2)$. For example, with $\alpha = 0.30 \text{ fm}^{-1}$, $r_0 = 2.8303 \text{ fm}$ and the minimum potential is $V(r_0 = 2.8303 \text{ fm}) = 15.746 \text{ fm}^{-1}$. It is worth noting that, in the limiting case when $\alpha \to 0$, the trigonometric PT potential can be reduced to the Kratzer potential [22, 23] $V(r) = D_e[(r - r_e)/r]^2 + \eta$, where $r_e$ is the equilibrium intermolecular separation and $D_e$ is the dissociation energy between diatomic molecules. In our case, $D_e = V_1$, $\eta = V_2$ and $r_e = 1/\alpha$.

In the case where $\eta = 0$, it reduces to the molecular potential, which is called the modified Kratzer potential proposed by Simons \textit{et al.} [24] and

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Molski and Konarski [25]. In the case where $\eta = -D_e$, this potential turns into the Kratzer potential, which includes an attractive Coulomb potential and a repulsive inverse square potential, introduced by Kratzer in 1920 [26].

The aim of the present work is to extend our previous work [21] to the case of $l \neq 0$ (rotational case). We introduce a convenient approximation scheme to deal with the strong singular centrifugal term.

The ansatz of this approximation possesses the same form of the potential and is singular as the centrifugal term $r^{-2}$. Thus, the Schrödinger equation with the trigonometric PT potential is solved approximately for its energy eigenvalues and corresponding wavefunctions with an arbitrary rotation–vibration $(n,l)$ state [27].

The paper is organized as follows. In Section 2, the NU method with all the necessary formulae used in the

![Figure 1](image-url)
calculations are briefly introduced and a shortcut of the method is given in Appendix A. In Section 3 we solve the SE for the given trigonometric PT potential and obtain its energy eigenvalues and the corresponding wavefunctions. Numerical results are obtained for any arbitrary vibration–rotation quantum numbers \( n \) and \( l \). Finally, the conclusion is given in Section 4.

2. The NU method

The NU method can be used to solve second-order differential equations with an appropriate coordinate transformation \( s = s(r) \) [28]

\[
\psi''_n(s) + \frac{\tilde{\tau}(s)}{\tilde{\sigma}(s)} \psi'_n(s) + \frac{\tilde{\sigma}(s)}{\tilde{\sigma}(s)} \psi_n(s) = 0,
\]

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at most of second degree, and \( \tilde{\tau}(s) \) is a first-degree polynomial. To find a particular solution of Equation (4), using the separation of variables, one can insert the transformation \( \psi_n(s) = \phi(s) y_n(s) \) to reduce the above equation to the form of the following hypergeometric type:

\[
\sigma(s) y''_n(s) + \tau(s) y'_n(s) + \lambda y_n(s) = 0.
\]

Furthermore, the function \( \phi(s) \) is defined by the logarithmic derivative

\[
\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}.
\]

The second part function \( y_n(s) \) is in the form of a hypergeometric-type function whose solutions are given by Rodrigues relation

\[
y_n(s) = B_n \frac{d^n}{d\sigma^n} [\sigma^\prime(s)\rho(s)],
\]

where \( B_n \) is the normalization constant and \( \rho(s) \) is the weight function that satisfies the condition [28]

\[
\frac{d}{ds} w(s) = \frac{\pi(s)}{\sigma(s)} w(s), \quad w(s) = \sigma(s)\rho(s).
\]

The function \( \pi(s) \) and the parameter \( \lambda \), required for this method, are defined as follows:

\[
\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left( \frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma},
\]

\[
\lambda = k + \pi'(s).
\]

In order to find the value of \( k \), the expression under the square root must be the square of a polynomial. Thus, the new eigenvalue equation is

\[
\lambda = \lambda_n = -n\tau' - \frac{n(n - 1)}{2}\sigma'',
\]

where

\[
\tau(s) = \tilde{\tau}(s) + 2\pi(s),
\]

and its derivative must be negative [28]. In this regard, one can also derive the parametric generalization version of the NU method [29] as displayed in Appendix A.

3. Solution of the radial SE for the trigonometric Pöschl–Teller potential

To study any quantum physical model characterized by the empirical molecular potential given by Equation (1), we need to solve the following SE [1, 2]:

\[
\left( \frac{\rho^2}{2m} + V(r) \right) \psi_{n,l,m}(r, \theta, \varphi) = E_{nl}\psi_{n,l,m}(r, \theta, \varphi),
\]

where the potential \( V(r) \) is taken as the trigonometric PT potential (1). Using the separation of variables by applying the wavefunction \( \psi(r, \theta, \varphi) = (1/r) R_n(r) \times Y_{l,m}(\theta, \varphi) \), we obtain the radial SE as

\[
\left[ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left( E_{nl} - \frac{V_1}{\sin^2(ar)} - \frac{V_2}{\cos^2(ar)} \right) - \frac{l(l+1)}{r^2} \right] R_n(r) = 0, \quad r > 0,
\]

where the radial wavefunction \( R_n(r) \) has to satisfy the required boundary conditions, namely \( R_n(0) = 0 \) and \( R_n(\pi/2) = 0 \), on the edges. Since the SE with the trigonometric PT potential has no analytical solution for \( l \neq 0 \) states, we resort to using an appropriate approximation scheme to deal with the centrifugal potential term

\[
\frac{1}{r^2} = \lim_{a \to 0} \alpha^2 \left( d_0 + \frac{1}{\sin^2(ar)} \right), \quad 0 < \alpha < \pi/2,
\]

where \( d_0 = 1/12 \) is a dimensionless shifting parameter and \( ar \ll 1 \). The approximation (14) is done on the basis that \( \sin(z) \approx z - z^3/3! + z^5/5! - z^7/7! + \cdots \), and in the limit when \( z \to 0 \), \( \sin(z) \approx z \). To demonstrate the validity and accuracy of our choice of the approximation scheme (14), we plot the centrifugal potential term \( 1/r^2 \) and its approximations, \( \alpha^2/\sin^2(ar) \) and \( \alpha^2(d_0 + 1/\sin^2(ar)) \), in Figure 2. As illustrated, the three curves coincide and show how accurate this replacement is. One of the present authors has treated this problem in recent work [30]. The insertion of
approximation (14) into Equation (13) gives

\[
\frac{d^2}{dr^2} + \frac{\varepsilon_{nl}}{\sin^2(\alpha r)} - \frac{V'_1}{\cos^2(\alpha r)} - l(l+1)\alpha^2 \\
\times \left(d_0 + \frac{1}{\sin^2(\alpha r)}\right) R_{n,l} = 0,
\]

where \( R_{n,l}(s) \simeq R_{n,l}(r) \) and also we have defined

\[
A = \frac{1}{4\alpha^2} (\varepsilon_{nl} - l(l+1)\alpha^2d_0),
\]

\[
B = \frac{1}{4\alpha^2} (\varepsilon + V'_1 - V'_2 + l(l+1)\alpha^2(1-C_0)),
\]

\[
C = \frac{1}{4\alpha^2} (V'_1 + l(l+1)\alpha^2).
\]

Comparing Equation (16) and relation (A2), we can easily obtain the coefficients \( c_i \) \( (i = 1, 2, 3) \) as follows:

\[
c_1 = \frac{1}{2}, \quad c_2 = 1, \quad c_3 = 1.
\]

The values of the remaining coefficients \( c_i \) \( (i = 4, 5, \ldots, 13) \) are found from relation (A5) of Appendix A. In addition, the specific values of the coefficients \( c_i \) \( (i = 1, 2, \ldots, 13) \) are listed in Table 1.

Using relation (A10), we can obtain the energy eigenvalues of the rotating trigonometric PT potential

\[
E_{nl} = \frac{\hbar^2 \alpha^2 l(l+1)d_0}{2m} + \frac{2\hbar^2 \alpha^2}{m} \left[ n + \frac{1}{2} - \frac{1}{4} \right] \left( \sqrt{2l+1} + \frac{8mV_1}{\hbar^2 \alpha^2} + \sqrt{1 + \frac{8mV_2}{\hbar^2 \alpha^2}} \right)^2.
\]

In the limit when \( \alpha \to 0 \), the energy formula (19) reduces to a constant value:

\[
\lim_{\alpha \to 0} E_{nl} = (\sqrt{V_1} + \sqrt{V_2})^2.
\]

Further, there is no loss of generality if \( d_0 = 0 \), then Equation (19) becomes

\[
E_{nl} = \frac{2\hbar^2 \alpha^2}{m} \left[ n + \frac{1}{2} + \frac{1}{4} \left( \sqrt{2l+1} + \frac{8mV_1}{\hbar^2 \alpha^2} + \sqrt{1 + \frac{8mV_2}{\hbar^2 \alpha^2}} \right) \right]^2.
\]

Figure 2. The centrifugal term \( 1/r^2 \) (green line) and its approximations (14).
where \( n = 0, 1, 2, \ldots \) and \( l = 0, 1, 2, \ldots \) are the vibration and rotation quantum numbers, respectively. To obtain numerical energy eigenvalues for the present potential model, we take the following set of parameter values, namely \( m = 10 \text{ fm}^{-1}, V_1 = 5.0 \text{ fm}^{-1}, V_2 = 3.0 \text{ fm}^{-1} \) and \( \alpha = 1.2, 0.8, 0.4, 0.2, 0.02, 0.002 \) [20]. As can be seen from Tables 2 and 3, in the limit when the potential range parameter \( \alpha \) approaches zero, the energy eigenvalues approach a constant value given by Equation (20). In Tables 2 and 3, we take \( d_0 = 1/12 \) and \( d_0 = 0 \), respectively. Figure 3 shows the variation of the lowest vibration–rotation \( 1s \), \( 1p \), \( 2s \), \( 2p \), \( 3s \) and \( 3p \) states with the screening parameter \( \alpha \) for the set of parameter values \( m = 10 \text{ fm}^{-1}, V_1 = 5.0 \text{ fm}^{-1}, V_2 = 3.0 \text{ fm}^{-1} \) and \( d_0 = 0 \). Further, for the same set of parameters, we show the energy states versus the vibration quantum number \( n \) in Figure 4.

We now need to calculate the wavefunctions. Using Table 1 together with relations (A11) and (A12) of Appendix A, we obtain the functions

\[
\rho(s) = s^{(1/2)}\sqrt{(2l+1)^2 + (8mV_1/h^2 \omega^2)}(1 - s)^{(1/2)}\sqrt{1 + (8mV_2/h^2 \omega^2)} ,
\]

(22)

\[
\phi(s) = s^{(1/4)}\left[1 + \sqrt{(2l+1)^2 + (8mV_1/h^2 \omega^2)}\right](1 - s)^{(1/4)}\left[1 + \sqrt{1 + (8mV_2/h^2 \omega^2)}\right],
\]

(23)

Further, relation (A13) gives the first part of the desired wavefunction:

\[
y_n(s) = P_n^{(1/2)}\sqrt{(2l+1)^2 + (8mV_1/h^2 \omega^2)}(1/2)\sqrt{1 + (8mV_2/h^2 \omega^2)}(1 - 2s),
\]

(24)

and employing \( R_{nl}(s) = \phi(s) y_n(s) \), we finally obtain the radial wavefunctions from relation (A14) as

\[
R_{nl}(s) = s^{(1/4)}\left[1 + \sqrt{(2l+1)^2 + (8mV_1/h^2 \omega^2)}\right]
\times (1 - s)^{(1/4)}\left[1 + \sqrt{1 + (8mV_2/h^2 \omega^2)}\right]
\times P_n^{(1/2)}\sqrt{(2l+1)^2 + (8mV_1/h^2 \omega^2)}(1/2)\sqrt{1 + (8mV_2/h^2 \omega^2)}
\times (1 - 2s).
\]

(25)

---

**Table 1.** Values of the coefficients used in energy and wavefunction calculations.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Numerical or analytic value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_4 )</td>
<td>1/4</td>
</tr>
<tr>
<td>( \alpha_5 )</td>
<td>-1/2</td>
</tr>
<tr>
<td>( \alpha_6 )</td>
<td>( \frac{1}{4} \left( 1 + \frac{2l}{\alpha} - l(l+1)d_0 \right) )</td>
</tr>
<tr>
<td>( \alpha_7 )</td>
<td>( -\frac{1}{4} \left[ (2l+1)^2 + \frac{2l}{\alpha} (e + V_1 - V_2) - l(l+1)d_0 \right] )</td>
</tr>
<tr>
<td>( \alpha_8 )</td>
<td>( \frac{1}{4\pi} \left( 1 + \frac{4l\alpha}{\omega^2} \right) )</td>
</tr>
<tr>
<td>( \alpha_9 )</td>
<td>( \frac{1}{4\pi} \left( 1 + \frac{4l\alpha}{\omega^2} \right) )</td>
</tr>
<tr>
<td>( \alpha_{10} )</td>
<td>( \frac{1}{2} \sqrt{(2l+1)^2 + \frac{4l\alpha}{\omega^2}} )</td>
</tr>
<tr>
<td>( \alpha_{11} )</td>
<td>( \frac{1}{2} \sqrt{1 + \frac{4l\alpha}{\omega^2}} )</td>
</tr>
<tr>
<td>( \alpha_{12} )</td>
<td>( \frac{1}{2} \left[ 1 + \left( \sqrt{(2l+1)^2 + \frac{4l\alpha}{\omega^2}} \right) \right] )</td>
</tr>
<tr>
<td>( \alpha_{13} )</td>
<td>( \frac{1}{4} \left( 1 + \sqrt{1 + \frac{4l\alpha}{\omega^2}} \right) )</td>
</tr>
</tbody>
</table>

---

**Table 2.** Bound state energy levels \( E_{nl} \) for the trigonometric PT potential.

<table>
<thead>
<tr>
<th>State ((n, l))</th>
<th>( \alpha = 1.2 )</th>
<th>( \alpha = 0.8 )</th>
<th>( \alpha = 0.4 )</th>
<th>( \alpha = 0.2 )</th>
<th>( \alpha = 0.02 )</th>
<th>( \alpha = 0.002 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2p</td>
<td>28.64395419</td>
<td>23.82847894</td>
<td>19.53712286</td>
<td>17.58054181</td>
<td>15.92402153</td>
<td>15.76317860</td>
</tr>
<tr>
<td>3s [21]</td>
<td>34.28835086</td>
<td>27.2944896</td>
<td>21.11625126</td>
<td>18.33059518</td>
<td>15.99541071</td>
<td>15.77082105</td>
</tr>
<tr>
<td>3p</td>
<td>34.67512504</td>
<td>27.44896381</td>
<td>21.15044543</td>
<td>18.3385626</td>
<td>15.9958560</td>
<td>15.77082179</td>
</tr>
<tr>
<td>3d</td>
<td>35.43921159</td>
<td>27.75631556</td>
<td>21.21875330</td>
<td>18.3545399</td>
<td>15.99563534</td>
<td>15.77082328</td>
</tr>
<tr>
<td>4p</td>
<td>41.28229854</td>
<td>31.32544868</td>
<td>22.82776800</td>
<td>19.11263070</td>
<td>16.06710967</td>
<td>15.77792658</td>
</tr>
<tr>
<td>4f</td>
<td>43.33519178</td>
<td>32.14003977</td>
<td>23.00470171</td>
<td>19.15336297</td>
<td>16.06748485</td>
<td>15.77793030</td>
</tr>
</tbody>
</table>
Inserting $s = \sin^2(\alpha r)$ into the above equation, we obtain

$$R_n(r) = N_n(\sin(\alpha r))^{(1+\eta_l)/2} \times (\cos(\alpha r))^{(1+\delta)/2} P_{\eta_l/2,\delta/2}^{(\eta_l/2,\delta/2)}(\cos(2\alpha r)), \quad (26a)$$

$$\eta_l = \sqrt{(2l + 1)^2 + (8mV_1/\hbar^2\alpha^2)}, \quad \delta = \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}}, \quad (26b)$$

where $N_n$ is a normalization factor to be calculated from the normalization conditions. For example, the ground s-state has the wavefunction

$$R_{00}(r) = N_{00}(\sin(\alpha r))^{(1+\eta_0)/2}(\cos(\alpha r))^{(1+\delta)/2} \times P_{\eta_0/2,\delta/2}^{(\eta_0/2,\delta/2)}(\cos(2\alpha r)), \quad (27a)$$

where

$$\eta_0 = \sqrt{1 + \frac{8mV_1}{\hbar^2\alpha^2}}.$$
and for the 1p-state

\[
R_{0,1}(r) = N_{01}(\sin(\alpha r))(1+\eta_{1}/2)^{(1+\delta)/2}(\cos(\alpha r))(1+\eta_{1}/2)^{(1-\delta)/2}
\times P_0^{(\eta_{1}/2,\delta/2)}(\cos(2\alpha r)),
\]

where

\[
\eta_{1} = \sqrt{9 + \frac{8mV_1}{\hbar^2\alpha^2}}.
\]
For illustration of this radial wavefunction, i.e. $R_n(r)$, of the trigonometric PT potential with various rotation–vibration 1s, 1p, 2s, 2p, 3s, and 3p states, see the curves in Figure 5. Clearly, the number of nodes (in the allowed range) increases with increasing vibration quantum number $n$. For example, the 1s and 1p states have one node, the 2s and 2p states have two nodes, etc. That is, the wavefunctions of the rotating trigonometric Pöschl–Teller oscillator potential increase their oscillations with increasing vibration quantum number $n$.

4. Final remarks and conclusion

In this work, we have obtained the approximate bound state solutions of the Schrödinger equation with the trigonometric Pöschl–Teller potential for an arbitrary $l$-state in the framework of a new approximation for the centrifugal term $r^{-2}$. We employed a shortcut of the NU method to determine the energy eigenvalues and corresponding wavefunctions. Numerical results are given in Tables 2 and 3 and it is found that, in the limit when the potential range parameter $\alpha \to 0$, the energy levels approach a constant value $\sqrt{V_1} + \sqrt{V_2}$. In the limiting case when $\alpha \to 0$, the trigonometric PT potential can be reduced to the Kratzer potential. We used a set of parameter values (listed in Table 4) to calculate the energy spectrum of the diatomic molecules I$_2$, LiH, HCl, O$_2$, H$_2$, NO and CO as illustrated in Table 5.

Acknowledgments

S.M. Ikhdair acknowledges the partial support of the Scientific and Technological Research Council of Turkey.

References

Appendix A: Parametric generalization of the NU method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation $s = s(r)$ [28].

$$\psi''_s(s) + \frac{\tilde{r}(s)}{\sigma(s)} \psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_s(s) = 0, \quad (A1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{r}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without the need to check the validity of the solution, we present a shortcut for the method. First, we write the general form of the Schrödinger-like equation (A1) in a more general form applicable to any potential as follows [29]:

$$\psi''_s(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)} \psi'_s(s) + \frac{A s^2 + B s - C}{s^2(1 - c_3 s)^2} \psi_s(s) = 0, \quad (A2)$$

satisfying the wavefunctions.

$$\psi_n(s) = \varphi(s) \nu_n(s). \quad (A3)$$

Comparing (A2) with its counterpart (A1), we obtain the following identifications:

\[ \tilde{r}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s), \]
\[ \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3, \quad (A4) \]

Following the NU method [28], we obtain the following shortcut procedure [29].

(i) The relevant constant:

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3),$$
$$c_6 = c_2 + 4, \quad c_7 = 2c_4c_5 - B,$$
$$c_8 = c_4^2 + C, \quad c_9 = c_3(c_7 + c_3c_8) + c_6,$$
$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_9} - 1 \neq -1,$$
$$c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3} \sqrt{c_9} > -1, \quad c_3 \neq 0,$$
$$c_{12} = c_4 + \sqrt{c_9} > 0,$$
$$c_{13} = -c_4 + \frac{1}{c_3} \sqrt{c_9} - c_5 > 0, \quad c_3 \neq 0. \quad (A5)$$

(ii) The essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - \left[ \sqrt{c_9} + c_3 \sqrt{c_9} s - \sqrt{c_9} \right], \quad (A6)$$
$$k = -(c_7 + 2c_3c_8) - 2\sqrt{c_9c_8}, \quad (A7)$$
$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - \left[ \sqrt{c_9} + c_3 \sqrt{c_9} s - \sqrt{c_9} \right], \quad (A8)$$
$$\tau'(s) = -2c_4 - 2\sqrt{c_9} + c_3 \sqrt{c_9} < 0. \quad (A9)$$

(iii) The energy equation:

$$c_2 n - (2n + 1)c_3 + (2n + 1)(\sqrt{c_9} + c_3 \sqrt{c_9} s + m(n - 1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_9c_8} = 0. \quad (A10)$$

(iv) The wavefunctions:

$$\rho(s) = s^{c_4}(1 - c_3 s)^{c_5}, \quad (A11)$$
$$\phi(s) = s^{c_7}(1 - c_3 s)^{c_8}, \quad c_{12} > 0, c_{13} > 0, \quad (A12)$$
$$\nu_s(s) = P_n^{(c_4, c_5)}(1 - 2c_3 s), \quad c_{10} > -1, c_{11} > -1, \quad (A13)$$
$$\psi_n(s) = N_n s^{c_4}(1 - c_3 s)^{c_5} P_n^{(c_4, c_5)}(1 - 2c_3 s), \quad (A14)$$

where $P_n^{(\mu, \nu)}(x)$, $\mu > -1$, $\nu > -1$ and $x \in [-1, 1]$ are Jacobi polynomials with.

$$P_n^{(\mu, \nu)}(1 - 2s) = \frac{(\alpha + 1)n!}{n!} 2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s), \quad (A15)$$

and $N_n$ is a normalization constant. Also, the above wavefunctions can be expressed in terms of the hypergeometric function as.

$$\psi_n(s) = N_n s^{c_4}(1 - c_3 s)^{c_5} 2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3), \quad (A16)$$

where $c_{12} > 0, c_{13} > 0$ and $s \in [0, 1/c_3], c_3 \neq 0.$