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Approximate spin and pseudospin solutions to the Dirac equation for the inversely quadratic Yukawa potential and tensor interaction

M Hamzavi[1], **S M Ikhdair**[2] **and B I Ita**[3]

[1] Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran
[2] Physics Department, Near East University, Nicosia, North Cyprus, Mersin 10, Turkey
[3] Theoretical Chemistry Group, Department of Pure and Applied Chemistry, University of Calabar, Nigeria

E-mail: majid.hamzavi@gmail.com

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Abstract

We approximately solve the Dirac equation for the inversely quadratic Yukawa potential including a Coulomb-like tensor potential with arbitrary spin–orbit coupling quantum number κ. In the framework of the spin and pseudospin (pspin) symmetry, we obtain the energy eigenvalue equation and the corresponding eigenfunctions in closed form by using the Nikiforov–Uvarov method. The numerical results show that the Coulomb-like tensor interaction removes degeneracies between spin and pseudospin state doublets.

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1. Introduction

Relativistic symmetries of the Dirac Hamiltonian was discovered many years ago. However, these symmetries have recently been recognized empirically in nuclear and hadronic spectroscopies [1]. Within the framework of the Dirac equation, the pseudospin (pspin) symmetry is used to feature deform nuclei and superdeformation to establish an effective shell-model [2–4]. The spin symmetry is relevant for mesons [5] and occurs when the difference of the scalar $S(r)$ and vector $V(r)$ potentials is constant, i.e. $\Delta(r) = C_s$, and the pspin symmetry occurs when the sum of the scalar and vector potentials is constant, i.e. $\Sigma(r) = C_{\mathrm{ps}}$ [6–7]. The pspin symmetry refers to quasi-degeneracy of single-nucleon doublets with non-relativistic quantum number $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l and j denote the single-nucleon radial, orbital and total angular quantum numbers, respectively [8, 9]. Further, the total angular momentum is $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is the pseudo-angular momentum and $\tilde{s}$ is the pspin angular momentum [10]. Recently, the tensor potentials were introduced into the Dirac equation with the substitution $\vec{p} \to \vec{p} - \mathrm{i}m\omega\beta \cdot \hat{r}\, U(r)$ and a spin–orbit coupling added to the Dirac Hamiltonian [11, 12]. Lisboa *et al* [13] have studied a generalized relativistic harmonic oscillator for spin-1/2 particles by considering a Dirac Hamiltonian that contains quadratic vector and scalar potentials together with a linear tensor potential under conditions of pspin and spin symmetry. Alberto *et al* [14] studied the contribution of the isoscalar tensor coupling to the realization of pspin symmetry in nuclei. Akçay [15] solved exactly the Dirac equation with scalar and vector quadratic potentials including a Coulomb-like tensor potential. He also solved exactly the Dirac equation for a linear and Coulomb-like term containing the tensor potential [16]. Also, Aydoğdu and Sever [17] obtained the exact solution to the Dirac equation for the pseudoharmonic potential in the presence of linear tensor potential under pspin symmetry and showed that tensor interactions remove all degeneracies between members of pspin doublets. Ikhdair and Sever [10] solved the Dirac equation approximately for the Hulthén potential including Coulomb-like tensor potential with an arbitrary spin–orbit coupling number κ under spin and pspin symmetry limits. Very recently, Hamzavi *et al* [18, 19] presented exact solutions to the Dirac equation for Mie-type potential and approximate solutions to the Dirac equation for Morse potential with a Coulomb-like tensor potential.

There are many works on the solutions to the Schrödinger, Klein–Gordon (KG) and Dirac equations for various types of potentials by different authors [20–38].

In this work, our aim is to solve the Dirac equation for the inversely quadratic Yukawa (IQY) potential in the presence of spin and pspin symmetries and by including a Coulomb-like tensor potential. The IQY potential takes the following form:

$$V(r) = -\frac{V_0}{r^2}e^{-2\alpha r}, \tag{1}$$

where α is the screening parameter and V_0 is the depth of the potential. A form of Yukawa potential has been used earlier by Taseli [39] in obtaining modified Laguerre basis for hydrogen-like systems. Also, Kermode *et al* [40] have used different forms of the Yukawa potential to obtain the effective range functions. But not much has been done in solving the IQY potential.

This paper is organized as follows. In section 2, we briefly introduce the Dirac equation with scalar and vector potentials with arbitrary spin–orbit coupling quantum number κ including tensor interaction under spin and pspin symmetry limits. The Nikiforov–Uvarov (NU) method is presented in section 3. The energy eigenvalue equations and corresponding eigenfunctions are obtained in section 4. In this section, some remarks and numerical results are also presented. Finally, our conclusion is given in section 5.

2. The Dirac equation with tensor coupling potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ (in units $\hbar = c = 1$) is

$$\left[\vec{\alpha}\cdot\vec{p} + \beta\,(M + S(r)) - i\beta\vec{\alpha}\cdot\hat{r}U(r)\right]\psi(\vec{r}) = [E - V(r)]\,\psi(\vec{r}), \tag{2}$$

where E is the relativistic binding energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3}$$

where I is the 2×2 unitary matrix and $\vec{\sigma}$ are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The eigenvalues of the spin–orbit coupling operator are $\kappa = (j+\frac{1}{2}) > 0$ and $\kappa = -(j+\frac{1}{2}) < 0$ for unaligned spin $j = l - \frac{1}{2}$ and the aligned spin $j = l + \frac{1}{2}$, respectively. The set (H^2, K, J^2, J_z) can be taken as the complete set of conservative quantities with $\vec{J}$ being the total angular momentum operator and $K = (\vec{\sigma}\cdot\vec{L} + 1)$ is the spin–orbit where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa}(\vec{r}) \\ g_{n\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)\,Y^{l}_{jm}(\theta,\varphi) \\ iG_{n\kappa}(r)\,Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{5}$$

where $f_{n\kappa}(\vec{r})$ is the upper (large) component and $g_{n\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y^{l}_{jm}(\theta,\varphi)$ and $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ are spin and pspin spherical harmonics, respectively, and m is the projection of the angular momentum on the z-axis. Substituting equation (5) into equation (2) and making use of the following relations:

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \tag{6a}$$

$$(\vec{\sigma}\cdot\vec{P}) = \vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P} + i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{6b}$$

together with the properties

$$\begin{aligned}
(\vec{\sigma}\cdot\vec{L})Y^{\tilde{l}}_{jm}(\theta,\phi) &= (\kappa - 1)\,Y^{\tilde{l}}_{jm}(\theta,\phi), \\
(\vec{\sigma}\cdot\vec{L})Y^{l}_{jm}(\theta,\phi) &= -(\kappa - 1)\,Y^{l}_{jm}(\theta,\phi), \\
(\vec{\sigma}\cdot\hat{r})Y^{\tilde{l}}_{jm}(\theta,\phi) &= -Y^{l}_{jm}(\theta,\phi), \\
(\vec{\sigma}\cdot\hat{r})Y^{l}_{jm}(\theta,\phi) &= -Y^{\tilde{l}}_{jm}(\theta,\phi),
\end{aligned} \tag{7}$$

one obtains two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\,G_{n\kappa}(r), \tag{8a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\,F_{n\kappa}(r), \tag{8b}$$

where

$$\Delta(r) = V(r) - S(r), \tag{9a}$$

$$\Sigma(r) = V(r) + S(r). \tag{9b}$$

After eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in equations (8), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components:

$$\begin{aligned}
&\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r)\right]F_{n\kappa}(r) \\
&+ \frac{\frac{d\Delta(r)}{dr}}{M + E_{n\kappa} - \Delta(r)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r) \\
&= [(M + E_{n\kappa} - \Delta(r))\,(M - E_{n\kappa} + \Sigma(r))]\,F_{n\kappa}(r),
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r)\right]G_{n\kappa}(r) \\
&+ \frac{\frac{d\Sigma(r)}{dr}}{M - E_{n\kappa} + \Sigma(r)}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r) \\
&= [(M + E_{n\kappa} - \Delta(r))\,(M - E_{n\kappa} + \Sigma(r))]\,G_{n\kappa}(r),
\end{aligned} \tag{11}$$

respectively, where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$. The quantum number κ is related to the quantum numbers for spin symmetry l and pspin symmetry $\tilde{l}$ as

$$\kappa=\begin{cases}-(l+1)=-(j+1/2)\ (s_{1/2},\ p_{3/2},\ \text{etc})\\ j=l+\frac{1}{2},\quad \text{aligned spin}\ (\kappa<0),\\ +l=+(j+1/2)\ (p_{1/2},\ d_{3/2},\ \text{etc})\\ j=l-\frac{1}{2},\quad \text{unaligned spin}\ (\kappa>0),\end{cases} \tag{12}$$

and the quasidegenerate doublet structure can be expressed in terms of a pspin angular momentum $\tilde{s}=1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa=\begin{cases}-\tilde{l}=-(j+1/2)\ (s_{1/2},\ p_{3/2},\ \text{etc})\\ j=\tilde{l}-\frac{1}{2},\quad \text{aligned pspin}\ (\kappa<0),\\ +(\tilde{l}+1)=+(j+1/2)\ (d_{3/2},\ f_{5/2},\ \text{etc})\\ j=\tilde{l}+\frac{1}{2},\quad \text{unaligned spin}\ (\kappa>0),\end{cases} \tag{13}$$

where $\kappa=\pm1,\pm2,\ldots$. For example, $(1s_{1/2},\ 0d_{3/2})$ and $(0p_{3/2},\ 0f_{5/2})$ can be considered as pspin doublets.

2.1. Pseudospin symmetry limit

Ginocchio [7] showed that there is a connection between pspin symmetry and near equality of the time component of a vector potential and the scalar potential, $V(r)\approx-S(r)$. After that, Meng *et al* [41, 42] derived that if $\frac{\mathrm{d}[V(r)+S(r)]}{\mathrm{d}r}=\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}=0$ or $\Sigma(r)=C_{ps}=\text{constant}$, then pspin symmetry is exact in the Dirac equation. Here, we are taking $\Delta(r)$ as the IQY potential (1) and the tensor potential as the Coulomb-like potential, i.e.

$$\Delta(r)=-\frac{V_0}{r^2}\mathrm{e}^{-2\alpha r}, \tag{14}$$

$$U(r)=-\frac{H}{r},\quad H=\frac{Z_aZ_be^2}{4\pi\varepsilon_0},\quad r\geqslant R_c, \tag{15}$$

where $R_c=7.78$ fm is the Coulomb radius and Z_a and Z_b denote the charges of the projectile a and the target nuclei b, respectively [10]. Under this symmetry, equation (11) is recast in the simple form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa H}{r^2}+\frac{H}{r^2}-\frac{H^2}{r^2}\right]G_{n\kappa}(r)=\left[\tilde{\gamma}\left(-\frac{V_0}{r^2}\mathrm{e}^{-2\alpha r}\right)+\tilde{\beta}^2\right]G_{n\kappa}(r), \tag{16}$$

where $\kappa=-\tilde{l}$ and $\kappa=\tilde{l}+1$ for $\kappa<0$ and $\kappa>0$, respectively. Also, we identified $\tilde{\gamma}=E_{n\kappa}-M-C_{ps}$ and $\tilde{\beta}^2=(M+E_{n\kappa})(M-E_{n\kappa}+C_{ps})$.

2.2. Spin symmetry limit

In the spin symmetry limit, $\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}=0$ or $\Delta(r)=C_s=$ constant [41, 42], with $\Sigma(r)$ taken as the IQY potential (1) and the Coulomb-like tensor potential. Thus, equation (10) is recast in the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}-\frac{2\kappa H}{r^2}-\frac{H}{r^2}-\frac{H^2}{r^2}\right]F_{n\kappa}(r)=\left[\gamma\left(-\frac{V_0}{r^2}\mathrm{e}^{-2\alpha r}\right)+\beta^2\right]F_{n\kappa}(r), \tag{17}$$

where $\kappa=l$ and $\kappa=-l-1$ for $\kappa<0$ and $\kappa>0$, respectively. Also, $\gamma=M+E_{n\kappa}-C_s$ and $\beta^2=(M-E_{n\kappa})(M+E_{n\kappa}-C_s)$.

Since the Dirac equation with the IQY potential has no exact solution, we use an approximation for the centrifugal term as [43–46]

$$\frac{1}{r^2}\approx4a^2\frac{\mathrm{e}^{-2ar}}{(1-\mathrm{e}^{-2ar})^2}. \tag{18}$$

Finally, for the solutions to equations (16) and (17) with the above approximation, we will employ the NU method, which is briefly introduced in the following section.

3. The Nikiforov–Uvarov method

This method can be used to solve second-order differential equations with an appropriate coordinate transformation $s=s(r)$ [47]:

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{19}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. A solution to equation (19) is found by a separation of variables, using the transformation $\psi_n(s)=\varphi(s)y_n(s)$. It reduces (19) into an equation of hypergeometric type

$$\sigma(s)y_n''(s)+\tau(s)y_n'(s)+\lambda y_n(s)=0. \tag{20}$$

$y_n(s)$ is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation:

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[\sigma^n(s)\rho(s)\right], \tag{21}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition [47]

$$\frac{\mathrm{d}}{\mathrm{d}s}w(s)=\frac{\tau(s)}{\sigma(s)}w(s),\quad w(s)=\sigma(s)\rho(s), \tag{22}$$

and $\varphi(s)$ is defined by its logarithmic derivative relation

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)}. \tag{23}$$

The function $\pi(s)$ and the parameter λ, required for this method, are defined as follows:

$$\pi(s)=\frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+k\sigma}, \tag{24a}$$

$$\lambda=k+\pi'(s). \tag{24b}$$

In order to find the value of k, the expression under the square root must be a square of a polynomial. Thus, a new eigenvalue equation is

$$\lambda=\lambda_n=-n\tau'-\frac{n(n-1)}{2}\sigma'', \tag{25}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s), \tag{26}$$

and its derivative must be negative [47]. In this regard, one can derive the parametric NU method [48, 49] outlined in some detail in the appendix.

4. Solutions to the Dirac equation

We will now solve the Dirac equation with the IQY potential and tensor potential by using the NU method.

4.1. The pseudospin symmetric case

To obtain the solution to equation (16), by using the transformation $s=\mathrm{e}^{-2\alpha r}$, we rewrite it as follows:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{1-s}{s(1-s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{1}{s^2(1-s)^2}\times\left(-\Lambda_\kappa(\Lambda_\kappa-1)s+\tilde{\gamma}V_0s^2-\frac{\tilde{\beta}}{4\alpha^2}(1-s)^2\right)\right]G_{n\kappa}=0, \tag{27}$$

where $\Lambda_\kappa=\kappa+H$. Comparing equations (27) with (A.1), we obtain

$$\alpha_1=1,\quad \xi_1=\frac{\tilde{\beta}}{4\alpha^2}-\tilde{\gamma}V_0,$$
$$\alpha_2=1,\quad \xi_2=-\Lambda_\kappa(\Lambda_\kappa-1)+\frac{2\tilde{\beta}}{4\alpha^2}, \tag{28}$$
$$\alpha_3=1,\quad \xi_3=\frac{\tilde{\beta}}{4\alpha^2}$$

and from (A.6)–(A.12), we further obtain

$$\alpha_4=0,\quad \alpha_5=-\frac{1}{2},$$
$$\alpha_6=\frac{1}{4}+\frac{\tilde{\beta}}{4\alpha^2}-\tilde{\gamma}V_0,\quad \alpha_7=\Lambda_\kappa(\Lambda_\kappa-1)-\frac{2\tilde{\beta}}{4\alpha^2}, \tag{29}$$
$$\alpha_8=\frac{\tilde{\beta}}{4\alpha^2},\quad \alpha_9=\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0,$$

In addition, the energy eigenvalue equation can be obtained by using the relation (A.17) as follows:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}+\sqrt{\frac{\tilde{\beta}^2}{4\alpha^2}}\right)^2=\frac{\tilde{\beta}^2}{4\alpha^2}-\tilde{\gamma}V_0. \tag{30}$$

By substituting the explicit forms of $\tilde{\gamma}$ and $\tilde{\beta}^2$ after equation (16) into equation (30), one can readily obtain the closed form for the energy formula. In the limiting case when the screening parameter $\alpha\to 0$ (low screening regime), the potential approximates as

$$V_{\mathrm{IQY}}(r)=-V_0\lim_{\alpha\to 0}\frac{\mathrm{e}^{-2\alpha r}}{r^2}\cong\frac{A}{r^2}-\frac{B}{r}+C,$$

where the potential parameters are defined as $A=-V_0$, $B=-2\alpha V_0$, $C=-2\alpha^2V_0$. This potential is well known as the Mie-type potential [18, 27]. The energy eigenvalue equation for this potential has recently been found in [27] as

$$\sqrt{(E_{n\kappa}-M-C_{\mathrm{ps}})C+(M+E_{n\kappa})(M-E_{n\kappa}+C_{\mathrm{ps}})}=\frac{\left(E_{n\kappa}-M-C_{\mathrm{ps}}\right)B}{1+2n+2\sqrt{\left(\kappa-\frac{1}{2}\right)^2+\left(E_{n\kappa}-M-C_{\mathrm{ps}}\right)A}}. \tag{31}$$

The special case when $A=C=0$ and $C_{\mathrm{ps}}=0$ yields the energy formula for the Coulomb-like potential, i.e. [27, 50]

$$E_{n\kappa}=-M\frac{4(n+\kappa)^2-B^2}{4(n+\kappa)^2+B^2}. \tag{32}$$

Furthermore, when $n\to\infty$, one obtains $E=-M$ (continuum states); that is, it shows that when n goes to infinity the energy solution to equation (30) becomes finite (i.e. the exact pspin symmetric case given by equation (38) of [50]).

On the other hand, to find the corresponding wave functions, referring to equation (29) and relations (A.18) and (A.22) of the appendix, we find the functions

$$\rho(s)=s^{\frac{\tilde{\beta}}{\alpha}}(1-s)^{2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}}, \tag{33}$$

$$\phi(s)=s^{\frac{\tilde{\beta}}{2\alpha}}(1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}}. \tag{34}$$

Hence, relation (A.19) gives

$$y_n(s)=P_n^{\left(\frac{\tilde{\beta}}{\alpha},2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}\right)}(1-2s). \tag{35}$$

By using $G_{n\kappa}(s)=\phi(s)y_n(s)$, we obtain the lower component of the Dirac spinor from relation (A.24) as

$$G_{n\kappa}(s)=\tilde{B}_{n\kappa}s^{\frac{\tilde{\beta}}{2\alpha}}(1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}}\times P_n^{\left(\frac{\tilde{\beta}}{\alpha},2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\tilde{\gamma}V_0}\right)}(1-2s), \tag{36}$$

where $\tilde{B}_{n\kappa}$ is the normalization constant. The upper component of the Dirac spinor can be calculated from equation (8*b*) as

$$F_{n\kappa}(r)=\frac{1}{M-E_{n\kappa}+C_{\mathrm{ps}}}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r), \tag{37}$$

where $E_{n\kappa}\neq M+C_{\mathrm{ps}}$ and with the exact pspin symmetry when $C_{\mathrm{ps}}=0$, only a negative energy solution exists. The finiteness of our solution requires that the two-components of the wave function be defined over the entire range, $r\in(0,\in\infty)$. However, in the pspin limit, if positive energy is

Table 1. The pspin symmetric bound state energy eigenvalues in units of fm^{-1} of the IQY potential for several values of n and κ.

$\tilde{l}$	$n, \kappa<0$	(l, j)	$E_{n,\kappa<0}$ $H=5$	$E_{n,\kappa<0}$ $H=0$	$n-1, \kappa>0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$ $H=5$	$E_{n-1,\kappa>0}$ $H=0$
1	1, −1	$1s_{1/2}$	−0.495 018	−0.491 129	0, 2	$0d_{3/2}$	−0.487 533	−0.491 129
2	1, −2	$1p_{3/2}$	−0.491 129	−0.487 533	0, 3	$0f_{5/2}$	−0.484 054	−0.487 533
3	1, −3	$1d_{5/2}$	−0.487 533	−0.484 054	0, 4	$0g_{7/2}$	−0.480 635	−0.484 054
4	1, −4	$1f_{7/2}$	−0.484 054	−0.480 635	0, 5	$0h_{9/2}$	−0.477 254	−0.480 635
1	2, −1	$2s_{1/2}$	−0.491 152	−0.487 539	1, 2	$2d_{3/2}$	−0.484 057	−0.487 539
2	2, −2	$2p_{3/2}$	−0.487 539	−0.484 057	1, 3	$1d_{3/2}$	−0.480 637	−0.484 057
3	2, −3	$2d_{5/2}$	−0.484 057	−0.480 637	1, 4	$1g_{7/2}$	−0.477 255	−0.480 637
4	2, −4	$2f_{7/2}$	−0.480 637	−0.477 255	1, 5	$1h_{9/2}$	−0.473 898	−0.477 255

Table 2. The spin symmetric bound state energy eigenvalues in units of fm^{-1} of the IQY potential for several values of n and κ.

l	$n, \kappa<0$	$(l, j=l+1/2)$	$E_{n,\kappa<0}$ $H=5$	$E_{n,\kappa<0}$ $H=0$	$n, \kappa>0$	$(l, j=l-1/2)$	$E_{n,\kappa>0}$ $H=5$	$E_{n,\kappa>0}$ $H=0$
1	0, −2	$0p_{3/2}$	1.000 000	0.994 385	0, 1	$0p_{1/2}$	0.990 029	0.994 385
2	0, −3	$0d_{5/2}$	0.994 385	0.990 029	0, 2	$0d_{3/2}$	0.985 992	0.990 029
3	0, −4	$0f_{7/2}$	0.990 029	0.985 992	0, 3	$0f_{5/2}$	0.982 086	0.985 992
4	0, −5	$0g_{9/2}$	0.985 992	0.982 086	0, 4	$0g_{7/2}$	0.978 249	0.982 086
1	1, −2	$1p_{3/2}$	0.994 367	0.990 023	1, 1	$1p_{1/2}$	0.985 988	0.990 023
2	1, −3	$1d_{5/2}$	0.990 023	0.985 988	1, 2	$1f_{3/2}$	0.982 084	0.985 988
3	1, −4	$1f_{7/2}$	0.985 988	0.982 084	1, 3	$1f_{5/2}$	0.978 247	0.982 084
4	1, −5	$1g_{9/2}$	0.982 084	0.978 247	1, 4	$1g_{7/2}$	0.974 455	0.978 247

chosen, the upper-spinor component of the wave function will be no longer defined as obviously seen in equation (37). Further, introducing the Coulomb-like tensor does not affect the negativity of the energy spectrum in the pspin limit, but the main contribution is just to removing the degeneracy of the spectrum.

Of course, the energy eigenvalue equation (30) admits two solutions (negative and positive); however, we choose the negative energy solution to make the wave function normalizable in the given range [50–53].

4.2. The spin symmetric case

To avoid repetition in the solution of equation (17), we follow the same procedures explained in section 4.1 and hence obtain the following energy eigenvalue equation:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\gamma V_0}+\sqrt{\frac{\beta^2}{4\alpha^2}}\right)^2=\frac{\beta^2}{4\alpha^2}-\gamma V_0, \tag{38}$$

and the corresponding wave functions for the upper Dirac spinor as

$$F_{n\kappa}=B_{n\kappa}s^{\frac{\beta}{2\alpha}}(1-s)^{\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\gamma V_0}}\times P_n^{\left(\frac{\beta}{\alpha},2\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\gamma V_0}\right)}(1-2s), \tag{39}$$

where $\eta_\kappa=\kappa+H+1$ and B_{nk} is the normalization constant. Finally, the lower-spinor component of the Dirac equation can be obtained via equation (8*a*) as

$$G_{n\kappa}(r)=\frac{1}{M+E_{n\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r), \tag{40}$$

where $E_{n\kappa}\neq -M+C_s$.

4.3. Some remarks and numerical results

The tensor potential generates a new spin–orbit centrifugal term $\Lambda(\Lambda\pm1)$ where $\Lambda=\Lambda_\kappa$ or η_κ. Some numerical results are given in tables 1 and 2, where we use the parameter values $M=5.0\,\mathrm{fm}^{-1}$, $V_0=1.0$, $C_{ps}=-5.5\,\mathrm{fm}^{-1}$ and $C_s=6.0\,\mathrm{fm}^{-1}$. In table 1, we consider the same set of pspin symmetry doublets: $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, ... Also, in table 2, we consider the same set of spin symmetry doublets: $(0p_{1/2}, 0p_{3/2})$, $(1d_{3/2}, 1d_{5/2})$, $(0f_{5/2}, 0f_{7/2})$, $(0g_{7/2}, 0g_{9/2})$, ... We see that the tensor interaction removes the degeneracy between two states in spin doublets and pspin doublets. When $H\neq0$, the energy levels of the spin (pspin) aligned states and spin (pspin) unaligned states move in the opposite directions. For example, in the pspin doublet $(1s_{1/2}, 0d_{3/2})$: when $H=0$, $E_{1,-1}=E_{1,2}=-0.491\,129\,\mathrm{fm}^{-1}$, but when $H=5.0$, $E_{1,-1}=-0.495\,018\,\mathrm{fm}^{-1}$ with $\kappa<0$ and $E_{1,2}=-0.487\,533\,\mathrm{fm}^{-1}$ with $\kappa>0$. Also, Aydoğdu and Sever [17] showed that the tensor interaction does not change the radial node structure of the upper and lower components of the Dirac spinor and affects the shape of the radial wave functions.

5. Conclusion

In this paper, under spin and pspin symmetry limits, we have obtained the approximate solutions to the Dirac equation for the IQY potential by using the NU method. Also, we extended the exact spin and pspin symmetric solutions of the IQY potential by including the Coulomb-like tensor potential in the form of $-H/r$. Some numerical results are included in tables 1 and 2. Obviously, the degeneracy between the members of doublet states in spin and pspin symmetries is removed by tensor interaction.

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Appendix. Parametric Nikiforov–Uvarov method

The following equation is a general form of the Schrödinger-like equation written for any potential [47]:

$$\left[\frac{d^2}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{[s(1-\alpha_3 s)]^2}\right]\psi_n(s)=0. \tag{A.1}$$

Comparing the above equation with equation (2), we obtain [48, 49]

$$\tilde{\tau}(s)=\alpha_1-\alpha_2 s, \tag{A.2}$$

$$\sigma(s)=s(1-\alpha_3 s) \tag{A.3}$$

and

$$\tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3. \tag{A.4}$$

Further, substituting relations (2)–(4) into equation (7*a*), we find that

$$\pi(s)=\alpha_4+\alpha_5 s\pm\left[(\alpha_6-k\alpha_3)s^2+(\alpha_7+k)s+\alpha_8\right]^{1/2}, \tag{A.5}$$

where

$$\alpha_4=\tfrac{1}{2}(1-\alpha_1), \tag{A.6}$$

$$\alpha_5=\tfrac{1}{2}(\alpha_2-2\alpha_3), \tag{A.7}$$

$$\alpha_6=\alpha_5^2+\xi_1, \tag{A.8}$$

$$\alpha_7=2\alpha_4\alpha_5-\xi_2, \tag{A.9}$$

$$\alpha_8=\alpha_4^2+\xi_3. \tag{A.10}$$

We require that the function under the square root of relation (A.5) be the square of a polynomial according to the NU method. Thus,

$$k_{1,2}=-(\alpha_7+2\alpha_3\alpha_8)\pm 2\sqrt{\alpha_8\alpha_9}, \tag{A.11}$$

where

$$\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6. \tag{A.12}$$

For each k, the following πs are obtained:

$$k=-(\alpha_7+2\alpha_3\alpha_8)-2\sqrt{\alpha_8\alpha_9} \tag{A.13}$$

and thus π becomes

$$\pi(s)=\alpha_4+\alpha_5 s-\left[\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)s-\sqrt{\alpha_8}\right]. \tag{A.14}$$

For the same k and from equation (9) and the relations (A.2) and (A.5), we obtain

$$\tau(s)=\alpha_1+2\alpha_4-(\alpha_2-2\alpha_5)s-2\left[\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)s-\sqrt{\alpha_8}\right] \tag{A.15}$$

and

$$\begin{aligned}\tau'(s)&=-(\alpha_2-2\alpha_5)-2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)\\&=-2\alpha_3-2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)<0.\end{aligned} \tag{A.16}$$

When (2) together with (15) and (16) are used, the following energy equation is derived:

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \tag{A.17}$$

This equation gives the energy spectrum of the desired problem. The wave function can be calculated according to the following procedures. The weight function is obtained via equation (6) as

$$\rho(s)=s^{\alpha_{10}-1}(1-\alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1}, \tag{A.18}$$

and consequently, after substitution into equation (5), we obtain

$$y_n(s)=P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \tag{A.19}$$

with

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8} \tag{A.20}$$

and

$$\alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right), \tag{A.21}$$

where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. Further, using equation (4), we obtain the second part of the wave function as

$$\phi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}. \tag{A.22}$$

Hence, the total wave function becomes

$$\psi(s)=\phi(s)y_n(s), \tag{A.23}$$

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s),\quad \alpha_3\neq 0. \tag{A.24}$$

Here, the constant parameters are defined by

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8} \tag{A.25}$$

and

$$\alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right). \tag{A.26}$$

In some problems $\alpha_3=0$ [48]; hence the wave functions turn into Laguerre polynomials:

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-\alpha_3)s=L_n^{\alpha_{10}-1}(\alpha_{11}s) \tag{A.27}$$

and

$$\lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=e^{\alpha_{13}s}. \tag{A.28}$$

Therefore, the solution given in equation (A.24) takes the form

$$\psi(s)=s^{\alpha_{12}}e^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{A.29}$$

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