# Approximate solutions of the Dirac equation for the Rosen–Morse potential including the spin-orbit centrifugal term

Sameer M. Ikhdair[a)]
*Department of Physics, Near East University, Nicosia, North Cyprus 922153, Turkey*

(Received 22 June 2009; accepted 18 December 2009; published online 17 February 2010)

We give the approximate analytic solutions of the Dirac equation for the Rosen–Morse potential including the spin-orbit centrifugal term. In the framework of the spin and pseudospin symmetry concept, we obtain the analytic bound state energy spectra and the corresponding two-component upper and lower spinors of the two Dirac particles, in closed form, by means of the Nikiforov–Uvarov method. The special cases of the $s$-wave $\kappa=\pm 1$ $(l=\tilde{l}=0)$ Rosen–Morse potential, the Eckart-type potential, the $PT$-symmetric Rosen–Morse potential, and the nonrelativistic limits are briefly studied. © *2010 American Institute of Physics.*
[doi:10.1063/1.3293759]

## I. INTRODUCTION

Within the framework of the Dirac equation, the spin symmetry arises if the magnitudes of the attractive scalar potential $S(r)$ and repulsive vector potential are nearly equal, $S(r)\sim V(r)$ in the nuclei [i.e., when the difference potential $\Delta(r)=V(r)-S(r)=C_s=\text{const}$]. However, the pseudospin symmetry occurs if $S(r)\sim -V(r)$ are nearly equal [i.e., when the sum potential $\Sigma(r)=V(r)+S(r)=C_{\text{ps}}=\text{const}$].[1–3] The spin symmetry is relevant for mesons.[4] The pseudospin symmetry concept has been applied to many systems in nuclear physics and related areas[2–7] and used to explain features of deformed nuclei,[8] the superdeformation,[9] and to establish an effective nuclear shell-model scheme.[5,6,10] The pseudospin symmetry introduced in nuclear theory refers to a quasidegeneracy of the single-nucleon doublets and can be characterized with the nonrelativistic quantum numbers $(n,l,j=l+1/2)$ and $(n-1,l+2,j=l+3/2)$, where $n$, $l$, and $j$ are the single-nucleon radial, orbital, and total angular momentum quantum numbers for a single particle, respectively.[5,6] The total angular momentum is given as $j=\tilde{l}+\tilde{s}$, where $\tilde{l}=l+1$ is a pseudoangular momentum and $\tilde{s}=1/2$ is a pseudospin angular momentum. In real nuclei, the pseudospin symmetry is only an approximation and the quality of approximation depends on the pseudocentrifugal potential and pseudospin orbital potential.[11] Alhaidari *et al.*[12] investigated in detail the physical interpretation on the three-dimensional (3D) Dirac equation in the context of spin symmetry limitation $\Delta(r)=0$ and pseudospin symmetry limitation $\Sigma(r)=0$.

Some authors applied the spin and pseudospin symmetry on several physical potentials, such as the harmonic oscillator,[12–15] the Woods–Saxon potential,[16] the Morse potential,[17,18] the Hulthén potential,[19] the Eckart potential,[20–22] the molecular diatomic three-parameter potential,[23] the Pöschl–Teller potential,[24] the Manning–Rosen potential,[25] and the generalized Morse potential.[26]

The exact solutions of the Dirac equation for the exponential-type potentials are possible only for the $s$-wave ($l=0$ case). However, for $l$-states an approximation scheme has to be used to deal with the centrifugal and pseudocentrifugal terms. Many authors used different methods to study the partially exactly solvable and exactly solvable Schrödinger, Klein–Gordon (KG), and Dirac equations in one-dimensional, 3D, and/or any $D$-dimensional cases for different potentials.[27–39] In

[a)]Electronic addresses: sikhdair@neu.edu.tr and sikhdair@gmail.com.

 

the context of spatially dependent mass, we have also used and applied a recently proposed approximation scheme[40] for the centrifugal term to find a quasiexact analytic bound state solution of the radial KG equation with spatially dependent effective mass for scalar and vector Hulthén potentials in any arbitrary dimension $D$ and orbital angular momentum quantum number $l$ within the framework of the Nikiforov–Uvarov (NU) method.[40–42]

Another physical potential is the Rosen–Morse potential[43] expressed in the form

$$V(r) = -V_1 \operatorname{sech}^2 \alpha r + V_2 \tanh \alpha r, \tag{1}$$

where $V_1$ and $V_2$ denote the depths of the potential and $\alpha$ is the range of the potential. This potential is useful for describing interatomic interaction of the linear molecules and is helpful for discussing polyatomic vibration energies such as the vibration states of $NH_3$ molecule.[43] It is shown that the Rosen–Morse potential and its $PT$-symmetric version are the special cases of the five-parameter exponential-type potential model.[44,45] The exact energy spectrum of the trigonometric Rosen–Morse potential has been investigated by using supersymmetric and improved quantization rule methods.[46,47]

Recently, many works have been done to solve the Dirac equation to obtain the energy equation and the two-component spinor wave functions. Jia *et al.*[48] employed an improved approximation scheme to deal with the pseudocentrifugal term to solve the Dirac equation with the generalized Pöschl–Teller potential for arbitrary spin-orbit quantum number $\kappa$. Zhang *et al.*[49] solved the Dirac equation with equal Scarf-type scalar and vector potentials by the method of the supersymmetric quantum mechanics (SUSYQM), shape invariance approach, and the alternative method. Zou *et al.*[50] solved the Dirac equation with equal Eckart scalar and vector potentials in terms of SUSYQM method, shape invariance approach, and function analysis method. Wei and Dong[51] obtained approximately the analytical bound state solutions of the Dirac equation with the Manning–Rosen for arbitrary spin-orbit coupling quantum number $\kappa$. Thylwe[52] presented the approach inspired by amplitude-phase method for analyzing the radial Dirac equation to calculate phase shifts by including the spin and pseudospin symmetries of the relativistic spectra. Alhaidari[53] solved the Dirac equation by separation of variables in spherical coordinates for a large class of noncentral electromagnetic potentials. Berkdemir and Sever[54] investigated systematically the pseudospin symmetry solution of the Dirac equation for spin-1/2 particles moving within the Kratzer potential connected with an angle-dependent potential. Alberto *et al.*[55] concluded that the values of energy spectra may not depend on the spinor structure of the particle, i.e., whether one has a spin-1/2 or a spin 0 particle. Also, they showed that a spin-1/2 or a spin 0 particle with the same mass and subject to the same scalar $S(r)$ and vector $V(r)$ potentials of equal magnitude, i.e., $S(r) = \pm V(r)$, will have the same energy spectrum (isospectrality), including both bound and scattering states.

In the present paper, our aim is to study the analytic solutions of the Dirac equation for the Rosen–Morse potential with arbitrary spin-orbit quantum number $\kappa$ by using a new approximation to deal with the centrifugal term. However, we use the approximation given in Ref. 56, which is quite different from the ones used in our previous works,[39,40,42] $1/r^2 \approx \alpha^2[d + e^{-\alpha r}/(1-e^{-\alpha r})^2]$, where $d=0$ or $d=\frac{1}{12}$. The approximation given in Ref. 56 is convenient for the Rosen–Morse-type potential because one may propose a more reasonable physical wave function for this system. Under the conditions of the spin symmetry $S(r) \sim V(r)$ and pseudospin symmetry $S(r) \sim -V(r)$, we investigate the bound state energy eigenvalues and the corresponding upper- and lower-spinor wave functions in the framework of the NU method. We also show that the spin and pseudospin symmetry Dirac solutions can be reduced to the $S(r)=V(r)$ and $S(r)=-V(r)$ in the cases of exact spin symmetry limitation $\Delta(r)=0$ and pseudospin symmetry limitation $\Sigma(r)=0$, respectively. Furthermore, the solutions obtained for the Dirac equation can be easily reduced to the Schrödinger solutions when the appropriate map of parameters is used.

This paper is structured as follows. In Sec. II, we outline the NU method. Section III is devoted to the analytic bound state solutions of the (3+1)-dimensional Dirac equation for the Rosen–Morse quantum system obtained by means of the NU method. The spin symmetry and pseudospin symmetry solutions are investigated. In Sec. IV, we study the cases $\kappa=\pm 1$ ($l=\tilde{l}=0$, i.e., $s$-wave), the Eckart-type potential, and the $PT$-symmetric Rosen–Morse potential. Finally, the relevant conclusions are given in Sec. V.

## II. NU METHOD

The NU method[41] is briefly outlined here. It was proposed to solve the second-order differential equation of hypergeometric type

$$\psi_n''(r) + \frac{\tilde{\tau}(r)}{\sigma(r)}\psi_n'(r) + \frac{\tilde{\sigma}(r)}{\sigma^2(r)}\psi_n(r) = 0, \tag{2}$$

where $\sigma(r)$ and $\tilde{\sigma}(r)$ are polynomials, at most, of second degree, and $\tilde{\tau}(r)$ is a first-degree polynomial. In order to find a particular solution for Eq. (2), let us decompose the wave function $\psi_n(r)$ as follows:

$$\psi_n(r) = \phi(r) y_n(r), \tag{3}$$

and use

$$[\sigma(r)\rho(r)]' = \tau(r)\rho(r) \tag{4}$$

to reduce Eq. (2) to the form

$$\sigma(r) y_n''(r) + \tau(r) y_n'(r) + \lambda y_n(r) = 0, \tag{5}$$

with

$$\tau(r) = \tilde{\tau}(r) + 2\pi(r), \qquad \tau'(r) < 0, \tag{6}$$

where the prime denotes the differentiation with respect to $r$. One is looking for a family of solutions corresponding to

$$\lambda = \lambda_n = -n\tau'(r) - \tfrac{1}{2}n(n-1)\sigma''(r), \quad n = 0,1,2,\ldots . \tag{7}$$

Here, $y_n(r)$ can be expressed in terms of the Rodrigues relation

$$y_n(r) = \frac{B_n}{\rho(r)}\frac{\mathrm{d}^n}{\mathrm{d}r^n}[\sigma^n(r)\rho(r)], \tag{8}$$

where $B_n$ is the normalization constant and the weight function $\rho(r)$ is the solution of the differential equation (4). The other part of the wave function (3) must satisfy the following logarithmic equation:

$$\frac{\phi'(r)}{\phi(r)} = \frac{\pi(r)}{\sigma(r)}. \tag{9}$$

By defining

$$k = \lambda - \pi'(r), \tag{10}$$

one obtains the polynomial

$$\pi(r) = \tfrac{1}{2}[\sigma'(r) - \tilde{\tau}(r)] \pm \sqrt{\tfrac{1}{4}[\sigma'(r) - \tilde{\tau}(r)]^2 - \tilde{\sigma}(r) + k\sigma(r)}, \tag{11}$$

where $\pi(r)$ is a parameter at most of the order of 1. The expression under the square root sign in the above equation can be arranged as a polynomial of second order where its discriminant is zero. Hence, an equation for $k$ is obtained. After solving such an equation, the $k$ values are determined through the NU method.

In this regard, we derive a parametric generalization version of the NU method valid for any solvable potential by the method. We begin by writing the hypergeometric equation in general parametric form as

$$[r(c_3 - c_4 r)]^2 \psi_n''(r) + [r(c_3 - c_4 r)(c_1 - c_2 r)]\psi_n'(r) + (-\xi_1 r^2 + \xi_2 r - \xi_3)\psi_n(r) = 0, \tag{12}$$

with

$$\tilde{\tau}(r) = c_1 - c_2 r, \tag{13}$$

$$\sigma(r) = r(c_3 - c_4 r), \tag{14}$$

$$\tilde{\sigma}(r) = -\xi_1 r^2 + \xi_2 r - \xi_3, \tag{15}$$

with the coefficients $c_i$ $(i=1,2,3,4)$ and the analytic expressions $\xi_j$ $(j=1,2,3)$. Furthermore, in comparing Eq. (12) with the counterpart Eq. (2), one obtains the appropriate analytic polynomials, energy equation, and wave functions together with the associated coefficients expressed in general parametric form, as displayed in Appendix A.

## III. ANALYTIC SOLUTION OF THE DIRAC–ROSEN–MORSE PROBLEM

In spherical coordinates, the Dirac equation for fermionic massive spin-1/2 particles interacting with arbitrary scalar potential $S(r)$ and the time-component $V(r)$ of a four-vector potential can be expressed as[26,57–60]

$$[c\alpha \cdot \mathbf{p} + \beta(Mc^2 + S(r)) + V(r) - E]\psi_{n\kappa}(\mathbf{r}) = 0, \quad \psi_{n\kappa}(\mathbf{r}) = \psi(r,\theta,\phi), \tag{16}$$

where $E$ is the relativistic energy of the system, $M$ is the mass of a particle, $\mathbf{p}=-i\hbar\nabla$ is the momentum operator, and $\alpha$ and $\beta$ are $4\times 4$ Dirac matrices, i.e.,

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17}$$

where $\mathbf{I}$ denotes the $2\times 2$ identity matrix and $\sigma_i$ are the three-vector Pauli spin matrices. For a spherical symmetrical nucleus, the total angular momentum operator of the nuclei $\mathbf{J}$ and spin-orbit matrix operator $\mathbf{K}=-\beta(\sigma\cdot\mathbf{L}+\mathbf{I})$ commute with the Dirac Hamiltonian, where $\mathbf{L}$ is the orbital angular momentum operator. The spinor wave functions can be classified according to the radial quantum number $n$ and the spin-orbit quantum number $\kappa$ and can be written using the Pauli–Dirac representation in the following forms:

$$\psi_{n\kappa}(\mathbf{r}) = \begin{pmatrix} f_{n\kappa}(\mathbf{r}) \\ g_{n\kappa}(\mathbf{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r) Y_{jm}^{l}(\theta,\phi) \\ iG_{n\kappa}(r) Y_{jm}^{\tilde{l}}(\theta,\phi) \end{pmatrix}, \tag{18}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper- and lower-spinor components, respectively, and $Y_{jm}^{l}(\theta,\phi)$ and $Y_{jm}^{\tilde{l}}(\theta,\phi)$ are the spherical harmonic functions coupled to the total angular momentum $j$ and its projection $m$ on the $z$ axis. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry $l$ and pseudospin symmetry $\tilde{l}$ refer to the upper- and lower-spinor components, respectively, for which $l(l+1)=\kappa(\kappa+1)$ and $\tilde{l}(\tilde{l}+1)=\kappa(\kappa-1)$. The quantum number $\kappa$ is related to the quantum numbers for spin symmetry $l$ and pseudospin symmetry $\tilde{l}$ as

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right) & (s_{1/2},\ p_{3/2},\ \text{etc.}),\quad j = l + \frac{1}{2}, \quad \text{aligned spin } (\kappa < 0) \\ +l = +\left(j+\frac{1}{2}\right) & (p_{1/2},\ d_{3/2},\ \text{etc.}),\quad j = l - \frac{1}{2}, \quad \text{unaligned spin } (\kappa > 0), \end{cases}$$

and the quasidegenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s}=1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa = \begin{cases} -\tilde{l} = -\left(j+\frac{1}{2}\right) & (s_{1/2},\ p_{3/2},\ \text{etc.}),\quad j = \tilde{l} - 1/2, \quad \text{aligned spin } (\kappa < 0) \\ +(\tilde{l}+1) = +\left(j+\frac{1}{2}\right) & (d_{3/2},\ f_{5/2},\ \text{etc.}),\quad j = \tilde{l} + 1/2, \quad \text{unaligned spin } (\kappa > 0), \end{cases}$$

where $\kappa=\pm 1,\pm 2,\ldots$. For example, $(1s_{1/2},0d_{3/2})$ and $(2p_{3/2},1f_{5/2})$ can be considered as pseudospin doublets.

Thus, substituting Eq. (18) into Eq. (16) leads to the following two radial coupled Dirac equations for the spinor components:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)F_{n\kappa}(r) = (Mc^2 + E_{n\kappa} - \Delta(r))G_{n\kappa}(r), \tag{19a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)G_{n\kappa}(r) = (Mc^2 - E_{n\kappa} + \Sigma(r))F_{n\kappa}(r), \tag{19b}$$

where $\Delta(r)=V(r)-S(r)$ and $\Sigma(r)=V(r)+S(r)$ are the difference and sum potentials, respectively.

Under the spin symmetry [i.e., $\Delta(r)=C_s=\text{const}$], one can eliminate $G_{n\kappa}(r)$ in Eq. (19a), with the aid of Eq. (19b), to obtain a second-order differential equation for the upper-spinor component as follows:[16,26]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{\hbar^2c^2}(Mc^2 + E_{n\kappa} - C_s)\Sigma(r)\right]F_{n\kappa}(r) = \frac{1}{\hbar^2c^2}(E_{n\kappa}^2 - M^2c^4 + C_s(Mc^2 - E_{n\kappa}))F_{n\kappa}(r), \tag{20}$$

where $\kappa(\kappa+1)=l(l+1)$, $\kappa=l$ for $\kappa<0$ and $\kappa=-(l+1)$ for $\kappa>0$. The spin symmetry energy eigenvalues depend on $n$ and $\kappa$, i.e., $E_{n\kappa}=E(n,\kappa(\kappa+1))$. For $l\neq 0$, the states with $j=l\pm 1/2$ are degenerate. Further, the lower-spinor component can be obtained from Eq. (19a) as

$$G_{n\kappa}(r) = \frac{1}{Mc^2 + E_{n\kappa} - C_s}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)F_{n\kappa}(r), \tag{21}$$

where $E_{n\kappa}\neq -Mc^2$, only real positive energy states exist when $C_s=0$ (exact spin symmetry).

On the other hand, under the pseudospin symmetry [i.e., $\Sigma(r)=C_{\text{ps}}=\text{const}$), one can eliminate $F_{n\kappa}(r)$ in Eq. (19b), with the aid of Eq. (19a), to obtain a second-order differential equation for the lower-spinor component as follows:[16,26]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\kappa(\kappa-1)}{r^2} - \frac{1}{\hbar^2c^2}(Mc^2 - E_{n\kappa} + C_{\text{ps}})\Delta(r)\right]G_{n\kappa}(r) = \frac{1}{\hbar^2c^2}(E_{n\kappa}^2 - M^2c^4 - C_{\text{ps}}(Mc^2 + E_{n\kappa}))G_{n\kappa}(r), \tag{22}$$

and the upper-spinor component $F_{n\kappa}(r)$ is obtained from Eq. (19b) as

$$F_{n\kappa}(r) = \frac{1}{Mc^2 - E_{n\kappa} + C_{\text{ps}}}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)G_{n\kappa}(r), \tag{23}$$

where $E_{n\kappa}\neq Mc^2$, only real negative energy states exist when $C_{\text{ps}}=0$ (exact pseudospin symmetry). From the above equations, the energy eigenvalues depend on the quantum numbers $n$ and $\kappa$, and also the pseudo-orbital angular quantum number $\tilde{l}$ according to $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$, which implies that $j=\tilde{l}\pm 1/2$ are degenerate for $\tilde{l}\neq 0$. The quantum condition is obtained from the finiteness of the solution at infinity and at the origin point, i.e., $F_{n\kappa}(0)=G_{n\kappa}(0)=0$ and $F_{n\kappa}(\infty)=G_{n\kappa}(\infty)=0$.

At this stage, we take the vector and scalar potentials in the form of Rosen–Morse potential model [see Eq. (1)]. Equations (20) and (22) can be solved exactly for $\kappa=0,-1$ and $\kappa=0,1$, respectively, because of the spin-orbit centrifugal and pseudocentrifugal terms. Therefore, to find approximate solution for the radial Dirac equation with the Rosen–Morse potential, we have to use an approximation for the spin-orbit centrifugal term. For values of $\kappa$ that are not large and vibrations of the small amplitude about the minimum, Lu[56] introduced an approximation to the centrifugal term near the minimum point $r=r_e$ as

$$\frac{1}{r^2}\approx\frac{1}{r_e^2}\left[D_0+D_1\frac{-\exp(-2\alpha r)}{1+\exp(-2\alpha r)}+D_2\left(\frac{-\exp(-2\alpha r)}{1+\exp(-2\alpha r)}\right)^2\right], \tag{24}$$

where

$$D_0=1-\left(\frac{1+\exp(-2\alpha r_e)}{2\alpha r_e}\right)^2\left(\frac{8\alpha r_e}{1+\exp(-2\alpha r_e)}-(3+2\alpha r_e)\right),$$

$$D_1=-2(\exp(2\alpha r_e)+1)\left[3\left(\frac{1+\exp(-2\alpha r_e)}{2\alpha r_e}\right)-(3+2\alpha r_e)\left(\frac{1+\exp(-2\alpha r_e)}{2\alpha r_e}\right)\right],$$

$$D_2=(\exp(2\alpha r_e)+1)^2\left(\frac{1+\exp(-2\alpha r_e)}{2\alpha r_e}\right)^2\left(3+2\alpha r_e-\frac{4\alpha r_e}{1+\exp(-2\alpha r_e)}\right), \tag{25}$$

and higher order terms are neglected.

### A. Spin symmetry solution of the Rosen–Morse problem

We take the sum potential in Eq. (20) as the Rosen–Morse potential model, i.e.,

$$\Sigma(r)=-4V_1\frac{\exp(-2\alpha r)}{(1+\exp(-2\alpha r))^2}+V_2\frac{(1-\exp(-2\alpha r))}{(1+\exp(-2\alpha r))}. \tag{26}$$

The choice of $\Sigma(r)=2V(r)\rightarrow V(r)$, as mentioned in Ref. 12, enables one to reduce the resulting relativistic solutions into their nonrelativistic limit under appropriate transformations.

Using the approximation given by Eq. (24) and introducing a new parameter change $z(r)=-\exp(-2\alpha r)$, this allows us to decompose the spin-symmetric Dirac equation (20) into the Schrödinger-type equation in the spherical coordinates for the upper-spinor component $F_{n\kappa}(r)$,

$$\left[\frac{d^2}{dz^2}+\frac{(1-z)}{z(1-z)}\frac{d}{dz}+\frac{(-\beta_1z^2+\beta_2z-\varepsilon_{n\kappa}^2)}{z^2(1-z)^2}\right]F_{n\kappa}(z)=0,\quad F_{n\kappa}(0)=F_{n\kappa}(1)=0, \tag{27}$$

with

$$\varepsilon_{n\kappa}=\frac{1}{2\alpha}\sqrt{\frac{\omega}{r_e^2}D_0+\frac{1}{\hbar^2c^2}(Mc^2+E_{n\kappa}-C_s)(Mc^2-E_{n\kappa}+V_2)}>0, \tag{28a}$$

$$\beta_1=\frac{1}{4\alpha^2}\left\{\frac{\omega}{r_e^2}(D_0-D_1+D_2)+\frac{1}{\hbar^2c^2}(Mc^2+E_{n\kappa}-C_s)(Mc^2-E_{n\kappa}-V_2)\right\}, \tag{28b}$$

TABLE I. The specific values for the parametric constants necessary for calculating the energy eigenvalues and eigenfunctions of the spin symmetry Dirac wave equation.

| Constant | Analytic value |
|---|---|
| $c_1$ | 1 |
| $c_2$ | 1 |
| $c_3$ | 1 |
| $c_4$ | 1 |
| $c_5$ | 0 |
| $c_6$ | $-\frac{1}{2}$ |
| $c_7$ | $\frac{1}{4}+\beta_1$ |
| $c_8$ | $-\beta_2$ |
| $c_9$ | $\varepsilon_{n\kappa}^2$ |
| $c_{10}$ | $(\delta+\frac{1}{2})^2$ |
| $c_{11}$ | $2\varepsilon_{n\kappa}$ |
| $c_{12}=c_{15}$ | $2\delta+1$ |
| $c_{13}$ | $\varepsilon_{n\kappa}$ |
| $c_{14}=c_{16}$ | $\delta+1$ |
| $\xi_1$ | $\beta_1$ |
| $\xi_2$ | $\beta_2$ |
| $\xi_3$ | $\varepsilon_{n\kappa}^2$ |

$$\beta_2=\frac{1}{4\alpha^2}\left\{\frac{\omega}{r_e^2}(2D_0-D_1)+\frac{2}{\hbar^2c^2}(Mc^2+E_{n\kappa}-C_s)(Mc^2-E_{n\kappa}-2V_1)\right\}, \tag{28c}$$

where $\omega=\kappa(\kappa+1)$.

In order to solve Eq. (27) by means of the NU method, we should compare it with Eq. (2) to obtain the following particular values for the parameters:

$$\tilde{\tau}(z)=1-z,\quad \sigma(z)=z(1-z),\quad \tilde{\sigma}(z)=-\beta_1z^2+\beta_2z-\varepsilon_{n\kappa}^2. \tag{29}$$

Comparing Eqs. (13)–(15) with Eq. (29), we can easily obtain the coefficients $c_i$ $(i=1,2,3,4)$ and the analytic expressions $\xi_j$ $(j=1,2,3)$. However, the values of the coefficients $c_i$ $(i=5,6,\ldots,16)$ are found from relations (A1)–(A5) of Appendix A. Therefore, the specific values of the coefficients $c_i$ $(i=1,2,\cdots,16)$ together with $\xi_j$ $(j=1,2,3)$ are displayed in Table I. From relations (A6) and (A7) of Appendix A together with the coefficients in Table I, the selected forms of $\pi(z)$ and $k$ take the following particular values:

$$\pi(z)=\varepsilon_{n\kappa}-(1+\varepsilon_{n\kappa}+\delta)z, \tag{30}$$

$$k=\beta_2-[2\varepsilon_{n\kappa}^2+(2\delta+1)\varepsilon_{n\kappa}], \tag{31}$$

respectively, where

$$\delta=\frac{1}{2}\left(-1-\sqrt{1+\frac{\omega D_2}{\alpha^2r_e^2}+\frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2+E_{n\kappa}-C_s)}\right) \tag{32}$$

for bound state solutions. According to the NU method, relations (A8) and (A9) of Appendix A give

$$\tau(z)=1+2\varepsilon_{n\kappa}-(3+2\varepsilon_{n\kappa}+2\delta)z,$$

$$\tau'(z) = -(3 + 2\varepsilon_{n\kappa} + 2\delta) < 0, \tag{33}$$

where prime denotes the derivative with respect to $z$. In addition, relation (A10) of Appendix A gives the energy equation for the Rosen–Morse potential in the Dirac theory as

$$\begin{aligned}(Mc^2 + E_{n\kappa} - C_s)(Mc^2 - E_{n\kappa} + V_2) &= -\frac{\omega D_0}{r_e^2}\hbar^2 c^2 \\ &+ \alpha^2\hbar^2 c^2\left[\frac{-\dfrac{V_2}{2\alpha^2\hbar^2 c^2}(Mc^2 + E_{n\kappa} - C_s) + \dfrac{\omega(-D_1 + D_2)}{4\alpha^2 r_e^2}}{(n+\delta+1)} - (n+\delta+1)\right]^2.\end{aligned} \tag{34}$$

Further, for the exact spin symmetry case, $V(r)=S(r)$ or $C_s=0$, we obtain

$$\begin{aligned}(Mc^2 + E_{n\kappa})(Mc^2 - E_{n\kappa} + V_2) &= -\frac{\omega D_0}{r_e^2}\hbar^2 c^2 \\ &+ \alpha^2\hbar^2 c^2\left[\frac{-\dfrac{V_2}{2\alpha^2\hbar^2 c^2}(Mc^2 + E_{n\kappa} - C_s) + \dfrac{\omega(-D_1 + D_2)}{4\alpha^2 r_e^2}}{(n+\tilde{\delta}+1)} - (n+\tilde{\delta}+1)\right]^2,\end{aligned} \tag{35}$$

with

$$\tilde{\delta} = \delta(C_s \to 0). \tag{36}$$

Let us now find the corresponding wave functions for this model. Referring to Table I and relations (A11) and (A12) of Appendix A, we find the functions

$$\rho(z) = z^{2\varepsilon_{n\kappa}}(1-z)^{2\delta+1}, \tag{37}$$

$$\phi(z) = z^{\varepsilon_{n\kappa}}(1-z)^{\delta+1}. \tag{38}$$

Hence, relation (A13) of Appendix A gives

$$y_n(z) = A_n z^{-2\varepsilon_{n\kappa}}(1-z)^{-(2\delta+1)}\frac{\mathrm{d}^n}{\mathrm{d}z^n}[z^{n+2\varepsilon_{n\kappa}}(1-z)^{n+2\delta+1}] \sim P_n^{(2\varepsilon_{n\kappa},2\delta+1)}(1-2z), \quad z \in [0,1], \tag{39}$$

where the Jacobi polynomial $P_n^{(\mu,\nu)}(x)$ is defined only for $\mu > -1$, $\nu > -1$, and for the argument $x \in [-1,+1]$. By using $F_{n\kappa}(z) = \phi(z)y_n(z)$, we get the radial upper-spinor wave functions from relation (A14) as

$$\begin{aligned}F_{n\kappa}(z) &= \mathcal{N}_{n\kappa} z^{\varepsilon_{n\kappa}}(1-z)^{\delta+1}P_n^{(2\varepsilon_{n\kappa},2\delta+1)}(1-2z) \\ &= \mathcal{N}_{n\kappa}(\exp(-2\alpha r))^{\varepsilon_{n\kappa}}(1-\exp(-2\alpha r))^{\delta+1} \\ &\quad \times {}_2F_1(-n, n+2(\varepsilon_{n\kappa}+\delta+1); 2\varepsilon_{n\kappa}+1; -\exp(-2\alpha r)).\end{aligned} \tag{40}$$

The above upper-spinor component satisfies the restriction condition for the bound states, i.e., $\delta > 0$ and $\varepsilon_{n\kappa} > 0$. The normalization constants $\mathcal{N}_{n\kappa}$ are calculated in Appendix B.

Before presenting the corresponding lower component $G_{n\kappa}(r)$, let us recall a recurrence relation of hypergeometric function, which is used to solve Eq. (21) and present the corresponding lower component $G_{n\kappa}(r)$,

$$\frac{\mathrm{d}}{\mathrm{d}z}[{}_2F_1(a;b;c;z)] = \left(\frac{ab}{c}\right){}_2F_1(a+1;b+1;c+1;z), \tag{41}$$

with which the corresponding lower component $G_{n\kappa}(r)$ can be obtained as follows:

$$\begin{aligned} G_{n\kappa}(r) = {} & \frac{\mathcal{N}_{n\kappa}(-\exp(-2\alpha r))^{\varepsilon_{n\kappa}}(1+\exp(-2\alpha r))^{\delta+1}}{(Mc^2+E_{n\kappa}-C_s)}\left[-2\alpha\varepsilon_{n\kappa} - \frac{2\alpha(\delta+1)\exp(-2\alpha r)}{(1+\exp(-2\alpha r))} + \frac{\kappa}{r}\right] \\ & \times {}_2F_1(-n, n+2(\varepsilon_{n\kappa}+\delta+1); 2\varepsilon_{n\kappa}+1; -\exp(-2\alpha r)) \\ & + \mathcal{N}_{n\kappa}\left[\frac{2\alpha n[n+2(\varepsilon_{n\kappa}+\delta+1)](-\exp(-2\alpha r))^{\varepsilon_{n\kappa}+1}(1+\exp(-2\alpha r))^{\delta+1}}{(2\varepsilon_{n\kappa}+1)(Mc^2+E_{n\kappa}-C_s)}\right] \\ & \times {}_2F_1\left(-n+1; n+2\left(\varepsilon_{n\kappa}+\delta+\frac{3}{2}\right); 2(\varepsilon_{n\kappa}+1); -\exp(-2\alpha r)\right), \end{aligned} \tag{42}$$

where $E_{n\kappa} \neq -Mc^2$ for exact spin symmetry. Here, it should be noted that the hypergeometric series ${}_2F_1(-n, n+2(\varepsilon_{n\kappa}+\delta+1); 2\varepsilon_{n\kappa}+1; -\exp(-2\alpha r))$ terminates for $n=0$ and thus does not diverge for all values of real parameters $\delta$ and $\varepsilon_{n\kappa}$.

For $C_s > Mc^2+E_{n\kappa}$ and $E_{n\kappa} < Mc^2+V_2$ or $C_s < Mc^2+E_{n\kappa}$ and $E_{n\kappa} > Mc^2+V_2$, we note that parameters given in Eq. (28a) turn to be imaginary, i.e., $\varepsilon_{n\kappa}^2 < 0$ in the $s$-state ($\kappa=-1$). As a result, the condition of the existing bound states are $\varepsilon_{n\kappa} > 0$ and $\delta > 0$, that is to say, in the case of $C_s < Mc^2+E_{n\kappa}$ and $E_{n\kappa} < Mc^2+V_2$, bound states do exist for some quantum number $\kappa$ such as the $s$-state ($\kappa=-1$). Of course, if these conditions are satisfied for the existing bound states, the energy equation and wave functions are the same as those given in Eqs. (34), (40), and (42).

### B. Pseudospin symmetry solution of the Rosen–Morse problem

Now taking the difference potential in Eq. (22) as the Rosen–Morse potential model, i.e.,

$$\Delta(r) = -4V_1\frac{\exp(-2\alpha r)}{(1+\exp(-2\alpha r))^2} + V_2\frac{(1-\exp(-2\alpha r))}{(1+\exp(-2\alpha r))}, \tag{43}$$

leads us to obtain a Schrödinger-type equation for the lower-spinor component $G_{n\kappa}(r)$,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{(1-z)}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{(-\tilde{\beta}_1 z^2 + \tilde{\beta}_2 z - \tilde{\varepsilon}_{n\kappa}^2)}{z^2(1-z)^2}\right]G_{n\kappa}(z) = 0, \quad G_{n\kappa}(0) = G_{n\kappa}(1) = 0, \tag{44}$$

where

$$\tilde{\varepsilon}_{n\kappa} = \frac{1}{2\alpha}\sqrt{\frac{\tilde{\omega}}{r_e^2}D_0 - \frac{1}{\hbar^2c^2}[E_{n\kappa}^2 - M^2c^4 - (Mc^2+E_{n\kappa})C_{\mathrm{ps}} + (Mc^2-E_{n\kappa}+C_{\mathrm{ps}})V_2]} > 0, \tag{45a}$$

$$\tilde{\beta}_1 = \frac{1}{4\alpha^2}\left\{\frac{\tilde{\omega}}{r_e^2}(D_0-D_1+D_2) - \frac{1}{\hbar^2c^2}[E_{n\kappa}^2 - M^2c^4 - (Mc^2+E_{n\kappa})C_{\mathrm{ps}} - (Mc^2-E_{n\kappa}+C_{\mathrm{ps}})V_2]\right\}, \tag{45b}$$

$$\tilde{\beta}_2 = \frac{1}{4\alpha^2}\left\{\frac{\tilde{\omega}}{r_e^2}(2D_0-D_1) - \frac{2}{\hbar^2c^2}[E_{n\kappa}^2 - M^2c^4 - (Mc^2+E_{n\kappa})C_{\mathrm{ps}} - 2(Mc^2-E_{n\kappa}+C_{\mathrm{ps}})V_1]\right\}, \tag{45c}$$

and $\tilde{\omega}=\kappa(\kappa-1)$. To avoid repetition in the solution of Eq. (44), a first inspection for the relationship between the present set of parameters $(\tilde{\varepsilon}_{n\kappa}, \tilde{\beta}_1, \tilde{\beta}_2)$ and the previous set $(\varepsilon_{n\kappa}, \beta_1, \beta_2)$ tells us that the negative energy solution for pseudospin symmetry, where $S(r)=-V(r)$, can be obtained directly from those of the positive energy solution above for spin symmetry using the parameter map[57–59]

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), \quad V(r) \to -V(r) \quad (\text{or } V_1 \to -V_1 \quad \text{and} \quad V_2 \to -V_2), \quad E_{n\kappa} \to -E_{n\kappa}, \quad \text{and} \quad C_s \to -C_{\text{ps}}. \tag{46}$$

Following the previous results with the above transformations, we finally arrive at the energy equation

$$(Mc^2 - E_{n\kappa} + C_{\text{ps}})(Mc^2 + E_{n\kappa} - V_2) = -\frac{\tilde{\omega}D_0}{r_e^2}\hbar^2c^2 + \alpha^2\hbar^2c^2\left[\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 - E_{n\kappa} + C_{\text{ps}}) + \frac{\tilde{\omega}(-D_1 + D_2)}{4\alpha^2 r_e^2}}{(n+\delta_1+1)} - (n+\delta_1+1)\right]^2, \tag{47}$$

where

$$\delta_1 = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{\tilde{\omega}D_2}{\alpha^2 r_e^2} - \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 - E_{n\kappa} + C_{\text{ps}})}\right). \tag{48}$$

By using $G_{n\kappa}(z)=\phi(z)y_n(z)$, we get the radial lower-spinor wave functions as

$$G_{n\kappa}(r) = \tilde{\mathcal{N}}_{n\kappa}(-\exp(-2\alpha r))^{\tilde{\varepsilon}_{n\kappa}}(1+\exp(-2\alpha r))^{\delta_1+1}P_n^{(2\tilde{\varepsilon}_{n\kappa},2\delta_1+1)}(1+2\exp(-2\alpha r)). \tag{49}$$

The above lower-spinor component satisfies the restriction condition for the bound states, i.e., $\delta_1>0$ and $\tilde{\varepsilon}_{n\kappa}>0$. The normalization constants $\tilde{\mathcal{N}}_{nl}$ are calculated in Appendix B.

## IV. DISCUSSIONS

In this section, we are going to study four special cases of the energy eigenvalues given by Eqs. (34) and (47) for the spin and pseudospin symmetries, respectively. First, let us study $s$-wave case $l=0$ $(\kappa=-1)$ and $\tilde{l}=0$ $(\kappa=1)$ case

$$(Mc^2 + E_{n,-1} - C_s)(Mc^2 - E_{n,-1} + V_2) = \alpha^2\hbar^2c^2\left[\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1} - C_s)}{n+\delta_2+1} + n + \delta_2 + 1\right]^2, \tag{50}$$

where

$$\delta_2 = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1} - C_s)}\right). \tag{51}$$

If one sets $C_s=0$ into Eq. (50) and $C_{\text{ps}}=0$ into Eq. (47), we obtain for spin and pseudospin symmetric Dirac theory,

$$(Mc^2 + E_{n,-1})(Mc^2 - E_{n,-1} + V_2) = \alpha^2\hbar^2c^2\left[\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1})}{n+\delta_{-1}+1} + n + \delta_{-1} + 1\right]^2, \tag{52}$$

$$\delta_{-1} = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1})}\right), \tag{53}$$

and

$$(Mc^2 - E_{n,+1})(Mc^2 + E_{n,+1} - V_2) = +\alpha^2\hbar^2c^2\left[-\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 - E_{n,+1})}{n + \delta_{+1} + 1} + n + \delta_{+1} + 1\right]^2, \tag{54}$$

$$\delta_{+1} = \frac{1}{2}\left(-1 - \sqrt{1 - \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 - E_{n,+1})}\right). \tag{55}$$

respectively. The above solutions for the $s$-wave are found to be identical for spin and pseudospin cases $S(r)=V(r)$ and $S(r)=-V(r)$, respectively.

Second, when we set $V_1 \to -V_1$ and $V_2 \to -V_2$, the potential reduces to the Eckart-type potential and energy eigenvalues are given by

$$(Mc^2 + E_{n,-1})(Mc^2 - E_{n,-1} - V_2) = \alpha^2\hbar^2c^2\left[-\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1})}{n + \delta_{-1} + 1} + n + \delta_{-1} + 1\right]^2, \tag{56}$$

$$\delta_{-1} = \frac{1}{2}\left(-1 - \sqrt{1 - \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 + E_{n,-1})}\right) \tag{57}$$

for spin symmetry and

$$(Mc^2 - E_{n,+1})(Mc^2 + E_{n,+1} + V_2) = +\alpha^2\hbar^2c^2\left[\frac{\frac{V_2}{2\alpha^2\hbar^2c^2}(Mc^2 - E_{n,+1})}{n + \delta_{+1} + 1} + n + \delta_{+1} + 1\right]^2, \tag{58}$$

$$\delta_{+1} = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{4V_1}{\alpha^2\hbar^2c^2}(Mc^2 - E_{n,+1})}\right) \tag{59}$$

for pseudospin symmetry.

Third, let us now discuss the nonrelativistic limit of the energy eigenvalues and wave functions of our solution. If we take $C_s=0$ and put $S(r)=V(r)=\Sigma(r)$, the nonrelativistic limit of energy equation (34) and wave function (40) under the following appropriate transformations $(Mc^2 + E_{n\kappa})/\hbar^2c^2 \to 2\mu/\hbar^2$ and $Mc^2 - E_{n\kappa} \to -E_{nl}$ (Refs. 26, 57, and 42) become

$$E_{nl} = V_2 + \frac{\omega D_0}{2\mu r_e^2}\hbar^2 - \frac{\hbar^2}{2\mu}\alpha^2\left[\frac{\frac{\mu}{\alpha^2\hbar^2}(2V_1 + V_2)\frac{\omega D_1}{4\alpha^2 r_e^2} + (n+1)^2 + (2n+1)\tilde{\delta}_0}{n + \tilde{\delta}_0 + 1}\right]^2, \tag{60}$$

with

$$\tilde{\delta}_0 = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{8\mu V_1}{\alpha^2\hbar^2} + \frac{\omega D_2}{\alpha^2 r_e^2}}\right), \tag{61}$$

and the associated wave functions are

$$F_{nl}(r)=\mathcal{N}_{nl}(-\exp(-2\alpha r))^{\varepsilon_{nl}}(1+\exp(-2\alpha r))^{1+\tilde{\delta}_0}P_n^{(2\varepsilon_{nl},2\tilde{\delta}_0+1)}(1+2\exp(-2\alpha r)), \tag{62}$$

where

$$\varepsilon_{nl}=\frac{1}{2\alpha}\sqrt{\frac{\omega}{r_e^2}D_0+\frac{2\mu}{\hbar^2}(V_2-E_{nl})}>0, \quad \omega=l(l+1), \tag{63}$$

which are identical to Ref. 25 in the solution of the Schrödinger equation. Finally, the Jacobi polynomials can be expressed in terms of the hypergeometric function as

$$P_n^{(\mu,\nu)}(1-2\exp(-2\alpha r))=\frac{(\mu+1)_n}{n!}\,{}_2F_1(-n,1+\mu+\nu+n;\mu+1;\exp(-2\alpha r)), \tag{64}$$

where $z\in[0,1]$, which lie within or on the boundary of the interval $[-1,1]$.

Fourth, if we choose $V_2\to iV_2$, the potential becomes the $PT$-symmetric Rosen–Morse potential, where $P$ denotes parity operator and $T$ denotes time reversal. For a potential $V(r)$, making the transformation of $r\to -r$ (or $r\to\xi-r$) and $i\to -i$, if we have the relation $V(-r)=V^*(r)$), the potential $V(r)$ is said to be $PT$-symmetric.[43] In this case we obtain for spin-symmetric Dirac equation

$$(Mc^2+E_{n\kappa})(Mc^2-E_{n\kappa}+iV_2)=-\frac{\kappa(\kappa+1)D_0}{r_e^2}\hbar^2c^2 + \alpha^2\hbar^2c^2\left[\frac{-\frac{iV_2}{2\alpha^2\hbar^2c^2}(Mc^2+E_{n\kappa}-C_s)+\frac{\kappa(\kappa+1)(-D_1+D_2)}{4\alpha^2r_e^2}}{(n+\tilde{\delta}+1)}-(n+\tilde{\delta}+1)\right]^2. \tag{65}$$

In the nonrelativistic limit, it turns to become

$$E_{nl}=iV_2+\frac{l(l+1)\hbar^2D_0}{2\mu r_e^2}-\frac{\hbar^2}{2\mu}\alpha^2\left[\frac{\frac{\mu}{\alpha^2\hbar^2}(2V_1+iV_2)\frac{l(l+1)D_1}{4\alpha^2r_e^2}+(n+1)^2+(2n+1)\tilde{\delta}_0}{n+\tilde{\delta}_0+1}\right]^2, \tag{66}$$

where real $V_1>0$, which is identical to the results in Ref. 25. If one sets $l=0$ in the above equation, the result is identical to that in Refs. 44 and 45.

## V. CONCLUSIONS

We have analytically obtained the energy spectra and the corresponding wave functions of the Dirac equation for the Rosen–Morse potential under the conditions of the spin symmetry and pseudospin symmetry in the context of the NU method. For any spin-orbit coupling centrifugal term $\kappa$, we have found the explicit expressions for energy eigenvalues and associated wave functions in closed form. The most stringent interesting result is that the present spin and pseudospin symmetry cases can be easily reduced to the KG solution once $S(r)=V(r)$ and $S(r)=-V(r)$ (i.e., $C_s=C_{\rm ps}=0$).[55] The resulting solutions of the wave functions are expressed in terms of the generalized Jacobi polynomials. Obviously, the relativistic solution can be reduced to its nonrelativistic limit by the choice of appropriate mapping transformations. Also, in the case when spin-orbit quantum number $\kappa=0$, the problem reduces to the $s$-wave solution. The $s$-wave Rosen–Morse, the Eckart-type potential, the $PT$-symmetric Rosen–Morse potential, and the nonrelativistic cases are briefly studied.

## ACKNOWLEDGMENTS

The partial support provided by the Scientific and Technological Research Council of Turkey (TÜBİTAK) is highly appreciated. The author thanks the anonymous kind referees and editors for the very constructive comments and suggestions.

## APPENDIX A: PARAMETRIC GENERALIZATION OF THE NU METHOD

Our systematical derivation holds for any potential form.

(i) The relevant coefficients $c_i$ $(i=5,6,\ldots,16)$ are given as follows:

$$c_5=\tfrac{1}{2}(c_3-c_1), \quad c_6=\tfrac{1}{2}(c_2-2c_4), \quad c_7=c_6^2+\xi_1, \tag{A1}$$

$$c_8=2c_5c_6-\xi_2, \quad c_9=c_5^2+\xi_3, \quad c_{10}=c_4(c_3c_8+c_4c_9)+c_3^2c_7, \tag{A2}$$

$$c_{11}=\frac{2}{c_3}\sqrt{c_9}, \quad c_{12}=\frac{2}{c_3c_4}\sqrt{c_{10}}, \tag{A3}$$

$$c_{13}=\frac{1}{c_3}(c_5+\sqrt{c_9}), \quad c_{14}=\frac{1}{c_3c_4}(\sqrt{c_{10}}-c_4c_5-c_3c_6), \tag{A4}$$

$$c_{15}=\frac{2}{c_3}\sqrt{c_{10}}, \quad c_{16}=\frac{1}{c_3}(\sqrt{c_{10}}-c_4c_5-c_3c_6). \tag{A5}$$

(ii) The analytic results for the key polynomials,

$$\pi(r)=c_5+\sqrt{c_9}-\frac{1}{c_3}(c_4\sqrt{c_9}+\sqrt{c_{10}}-c_3c_6)r, \tag{A6}$$

$$k=-\frac{1}{c_3^2}(c_3c_8+2c_4c_9+2\sqrt{c_9c_{10}}), \tag{A7}$$

$$\tau(r)=c_3+2\sqrt{c_9}-\frac{2}{c_3}(c_3c_4+c_4\sqrt{c_9}+\sqrt{c_{10}})r, \tag{A8}$$

$$\tau'(r)=-\frac{2}{c_3}(c_3c_4+c_4\sqrt{c_9}+\sqrt{c_{10}})<0. \tag{A9}$$

(iii) The energy equation,

$$c_2n-(2n+1)c_6+\frac{1}{c_3}(2n+1)(\sqrt{c_{10}}+c_4\sqrt{c_9})+n(n-1)c_4+\frac{1}{c_3^2}(c_3c_8+2c_4c_9+2\sqrt{c_9c_{10}})=0. \tag{A10}$$

(iv) The wave functions,

$$\rho(r)=r^{c_{11}}(c_3-c_4r)^{c_{12}}, \tag{A11}$$

$$\phi(r)=r^{c_{13}}(c_3-c_4r)^{c_{14}}, \quad c_{13}>0, \quad c_{14}>0, \tag{A12}$$

$$y_{n\kappa}(r) = P_n^{(c_{11},c_{12})}(c_3 - 2c_4 r), \quad c_{11} > -1, \quad c_{12} > -1, \quad r \in [(c_3-1)/2c_4,(1+c_3)/2c_4], \tag{A13}$$

$$\psi_{n\kappa}(r) = \phi(r) y_{n\kappa}(r) = \mathcal{N}_n r^{c_{13}}(c_3 - c_4 r)^{c_{14}} P_n^{(c_{11},c_{12})}(c_3 - 2c_4 r), \tag{A14}$$

where $P_n^{(a,b)}(c_3-2c_4 r)$ are the Jacobi polynomials and $\mathcal{N}_n$ is a normalizing factor.

When $c_4=0$, the Jacobi polynomial turns to be the generalized Laguerre polynomial and the constants relevant to this polynomial change are

$$\lim_{c_4\to 0} P_n^{(c_{11},c_{12})}(c_3 - 2c_4 r) = L_n^{c_{11}}(c_{15} r), \tag{A15}$$

$$\lim_{c_4\to 0} (c_3 - c_4 r)^{c_{14}} = \exp(-c_{16} r), \tag{A16}$$

$$\psi_{n\kappa}(r) = \mathcal{N}_n \exp(-c_{16} r) L_n^{c_{11}}(c_{15} r), \tag{A17}$$

where $L_n^{c_{11}}(c_{15} r)$ are the generalized Laguerre polynomials and $\mathcal{N}_n$ is a normalizing constant.

## APPENDIX B: NORMALIZATION OF THE RADIAL WAVE FUNCTION

In order to find the normalization factor $\mathcal{N}_{n\kappa}$, we start by writing the normalization condition,

$$\frac{\mathcal{N}_{n\kappa}^2}{2\alpha}\int_0^1 z^{2\varepsilon_{n\kappa}-1}(1-z)^{2\delta+2}[P_n^{(2\varepsilon_{n\kappa},2\delta+1)}(1-2z)]^2 \mathrm{d}z = 1. \tag{B1}$$

Unfortunately, there is no formula available to calculate this key integration. Nevertheless, we can find the explicit normalization constant $\mathcal{N}_{n\kappa}$. For this purpose, it is not difficult to obtain the results of the above integral by using the following formulas:[61–64]

$$\int_0^1 (1-s)^{\mu-1} s^{\nu-1} {}_2F_1(\alpha,\beta;\gamma;as)\mathrm{d}s = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_3F_2(\nu,\alpha,\beta;\mu+\nu;\gamma;a), \tag{B2}$$

and

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)} \frac{z^p}{p!}.$$

Hence, the normalization constants for the upper-spinor component are

$$\mathcal{N}_{n\kappa} = \left[ \frac{\Gamma(2\delta+3)\Gamma(2\varepsilon_{n\kappa}+1)}{2\alpha\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (n+2(1+\varepsilon_{n\kappa}+\delta))_m \Gamma(n+m)}{m!\,(m+2\varepsilon_{n\kappa})!\,\Gamma\left(m+2\left(\varepsilon_{n\kappa}+\delta+\frac{3}{2}\right)\right)} f_{n\kappa} \right]^{-1/2}, \tag{B3}$$

with

$$f_{n\kappa} = {}_3F_2\left(2\varepsilon_{n\kappa}+m,-n,n+2(1+\varepsilon_{n\kappa}+\delta);m+2\left(\varepsilon_{n\kappa}+\delta+\tfrac{3}{2}\right);1+2\varepsilon_{n\kappa};1\right), \tag{B4}$$

where $(x)_m = \Gamma(x+m)/\Gamma(x)$. Also, the normalization constants for the lower-spinor component are

$$\tilde{\mathcal{N}}_{n\kappa}=\left[\frac{\Gamma(2\delta_1+3)\Gamma(2\tilde{\varepsilon}_{n\kappa}+1)}{2\alpha\Gamma(n)}\sum_{m=0}^{\infty}\frac{(-1)^m(n+2(1+\tilde{\varepsilon}_{n\kappa}+\delta_1))_m\Gamma(n+m)}{m!\,(m+2\tilde{\varepsilon}_{n\kappa})!\,\Gamma\left(m+2\left(\tilde{\varepsilon}_{n\kappa}+\delta_1+\frac{3}{2}\right)\right)}g_{n\kappa}\right]^{-1/2}, \tag{B5}$$

with

$$g_{n\kappa}={}_3F_2\left(2\tilde{\varepsilon}_{n\kappa}+m,-n,n+2(1+\tilde{\varepsilon}_{n\kappa}+\delta_1);m+2\left(\tilde{\varepsilon}_{n\kappa}+\delta_1+\tfrac{3}{2}\right);1+2\tilde{\varepsilon}_{n\kappa};1\right). \tag{B6}$$


[1] J. N. Ginocchio, Phys. Rev. C **69**, 034318 (2004).
[2] J. N. Ginocchio, Phys. Rev. Lett. **78**, 436 (1997).
[3] J. N. Ginocchio, Phys. Rep. **414**, 165 (2005).
[4] P. R. Page, T. Goldman, and J. N. Ginocchio, Phys. Rev. Lett. **86**, 204 (2001).
[5] A. Arima, M. Harvey, and K. Shimizu, Phys. Lett. B **30**, 517 (1969).
[6] K. T. Hecht and A. Adler, Nucl. Phys. A. **137**, 129 (1969).
[7] J. N. Ginocchio and D. G. Madland, Phys. Rev. C **57**, 1167 (1998).
[8] A. Bohr, I. Hamarnoto, and B. R. Motelson, Phys. Scr. **26**, 267 (1982).
[9] J. Dudek, W. Nazarewicz, Z. Szymanski, and G. A. Leander, Phys. Rev. Lett. **59**, 1405 (1987).
[10] D. Troltenier, C. Bahri, and J. P. Draayer, Nucl. Phys. A. **586**, 53 (1995).
[11] J. Meng, K. Sugawara-Tanabe, S. Yamaji, P. Ring, and A. Arima, Phys. Rev. C **58**, R628 (1998).
[12] A. D. Alhaidari, H. Bahlouli, and A. Al-Hasan, Phys. Lett. A **349**, 87 (2006).
[13] J. N. Ginocchio, Phys. Rev. Lett. **95**, 252501 (2005).
[14] J.-Y. Guo, X.-Z. Fang, and F.-X. Xu, Nucl. Phys. A. **757**, 411 (2005).
[15] R. Lisboa, M. Malheiro, A. S. De Castro, P. Alberto, and M. Fiolhais, Phys. Rev. C **69**, 024319 (2004); A. S. de Castro, P. Alberto, R. Lisboa, and M. Malheiro, *ibid.* **73**, 054309 (2006).
[16] J.-Y. Guo and Z.-Q. Sheng, Phys. Lett. A **338**, 90 (2005); S. M. Ikhdair and R. Sever, "Approximate analytical solutions of the generalized Woods-Saxon potentials including the spin-orbit coupling term and spin symmetry," Cent. Eur. J. Phys. (to be published).
[17] W. C. Qiang, R. S. Zhou, and Y. Gao, J. Phys. A: Math. Theor. **40**, 1677 (2007).
[18] O. Bayrak and I. Boztosun, J. Phys. A: Math. Theor. **40**, 11119 (2007).
[19] A. Soylu, O. Bayrak, and I. Boztosun, J. Math. Phys. **48**, 082302 (2007).
[20] A. Soylu, O. Bayrak, and I. Boztosun, J. Phys. A: Math. Theor. **41**, 065308 (2008).
[21] C. S. Jia, P. Gao, and X. L. Peng, J. Phys. A **39**, 7737 (2006).
[22] L. H. Zhang, X. P. Li, and C. S. Jia, Phys. Lett. A **372**, 2201 (2008).
[23] C. S. Jia, J. Y. Liu, L. He, and L. Sun, Phys. Scr. **75**, 388 (2007).
[24] C. S. Jia, P. Gao, Y. F. Diao, L. Z. Yi, and X. J. Xie, Eur. Phys. J. A **34**, 41 (2007).
[25] F. Taşkın, Int. J. Theor. Phys. **48**, 1142 (2009).
[26] S. M. Ikhdair, "Solutions of the Dirac equation for the generalized Morse potential by Nikiforov-Uvarov method," Phys. Scr. (submitted).
[27] R. Koç and M. Koca, J. Phys. A **36**, 8105 (2003).
[28] C.-S. Jia and A. de Souza Dutra, Ann. Phys. **323**, 566 (2008); C. S. Jia, P. Q. Wang, J. Y. Liu, and S. He, Int. J. Theor. Phys. **47**, 2513 (2008).
[29] J. Yu, S.-H. Dong, and G.-H. Sun, Phys. Lett. A **322**, 290 (2004); C. Gang, *ibid.* **329**, 22 (2004); C. Berkdemir, Nucl. Phys. A. **770**, 32 (2006).
[30] Y. Xu, S. He, and C.-S. Jia, J. Phys. A: Math. Theor. **41**, 255302 (2008); H. Akcay, *ibid.* **42**, 198001 (2009).
[31] A. de Souza Dutra and C.-S. Jia, Phys. Lett. A **352**, 484 (2006).
[32] G. Chen and Z. D. Chen, Phys. Lett. A **331**, 312 (2004).
[33] R. Sever and C. Tezcan, Int. J. Mod. Phys. E **17**, 1327 (2008).
[34] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. C **20**, 361 (2009).
[35] A. de Souza Dutra and C. A. S. Almeida, Phys. Lett. A **275**, 25 (2000).
[36] A. D. Alhaidari, Phys. Lett. A **322**, 72 (2004).
[37] I. O. Vakarchuk, J. Phys. A **38**, 4727 (2005).
[38] A. Arda, R. Sever, and C. Tezcan, Phys. Scr. **79**, 015006 (2009).
[39] S. M. Ikhdair and R. Sever, Phys. Scr. **79**, 035002 (2009).
[40] S. M. Ikhdair, Eur. Phys. J. A **39**, 307 (2009); **40**, 143 (2009).
[41] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics* (Birkhäuser, Bassel, 1988).
[42] S. M. Ikhdair, Int. J. Mod. Phys. C **20**, 25 (2009); S. M. Ikhdair and R. Sever, J. Math. Chem. **42**, 461 (2007); Ann. Phys. **16**, 218 (2007); Int. J. Theor. Phys. **46**, 1643 (2007); Int. J. Mod. Phys. C **19**, 221 (2008); **19**, 1425 (2008); Int. J. Mod. Phys. E **17**, 1107 (2008); Ann. Phys. **18**, 189 (2009); J. Math. Chem. **41**, 329 (2007); **41**, 343 (2007); S. M. Ikhdair, Chin. J. Phys. (Taipei) **46**, 291 (2008); S. M. Ikhdair and R. Sever, Cent. Eur. J. Phys. **6**, 141 (2008); J. Math. Chem. **45**, 1137 (2009).
[43] N. Rosen and P. M. Morse, Phys. Rev. **42**, 210 (1932).
[44] C.-S. Jia, S.-C. Li, Y. Li, and L.-T. Sun, Phys. Lett. A **300**, 115 (2002).

[45] C.-S. Jia, Y. Li, Y. Sun, J.-Y. Liu, and L.-T. Sun, Phys. Lett. A **311**, 115 (2003).
[46] C. B. Compean and M. Kirchbach, J. Phys. A **39**, 547 (2006).
[47] Z. Q. Ma, A. Gonzalez-Cisneros, B.-W. Xu, and S. H. Dong, Phys. Lett. A **371**, 180 (2007).
[48] C.-S. Jia, T. Chen, and L.-G. Cui, Phys. Lett. A **373**, 1621 (2009).
[49] X.-C. Zhang, Q.-W. Liu, C.-S. Jia, and L.-Z. Wang, Phys. Lett. A **340**, 59 (2005).
[50] X. Zou, L.-Z. Yi, and C.-S. Jia, Phys. Lett. A **346**, 54 (2005).
[51] G. F. Wei and S. H. Dong, Phys. Lett. A **373**, 49 (2008).
[52] K.-E. Thylwe, Phys. Scr. **77**, 065005 (2008).
[53] A. D. Alhaidari, Ann. Phys. **320**, 453 (2005).
[54] C. Berkdemir and R. Sever, J. Phys. A: Math. Theor. **41**, 045302 (2008).
[55] P. Alberto, A. S. de Castro, and M. Malheiro, Phys. Rev. C **75**, 047303 (2007).
[56] J. Lu, Phys. Scr. **72**, 349 (2005).
[57] S.M. Ikhdair and R. Sever, "Solutions of the s-wave Dirac equation for a charged quantum harmonic oscillator in a uniform electric field," J. Math. Chem. (to be published).
[58] C. Berkdemir and Y.-F. Cheng, Phys. Scr. **79**, 035003 (2009).
[59] A. De Souza Dutra and M. Hott, Phys. Lett. A **356**, 215 (2006).
[60] W. Greiner, *Relativistic Quantum Mechanics* (Springer, Berlin, 1981).
[61] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th ed. (Academic, New York, 1994).
[62] G. Sezgo, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939).
[63] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
[64] S. M. Ikhdair, Int. J. Mod. Phys. C **20**, 1563 (2009); Chem. Phys. **361**, 9 (2009).