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Approximate relativistic bound states of a particle in Yukawa field with Coulomb tensor interaction

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Abstract
We obtain the approximate relativistic bound state of a spin-$\frac{1}{2}$ particle in the field of the Yukawa potential and a Coulomb-like tensor interaction with arbitrary spin–orbit coupling number κ under the spin and pseudospin (p-spin) symmetries. The asymptotic iteration method is used to obtain energy eigenvalues and the corresponding wave functions in their closed forms. Our numerical results show that the tensor interaction removes degeneracies between the spin and p-spin doublets and creates new degenerate doublets for various strengths of tensor coupling.

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1. Introduction

Within the framework of the Dirac equation, the pseudospin (p-spin) symmetry is used to feature deformed nuclei, superdeformation and to establish an effective shell model [1–4]. However, the spin symmetry is relevant for mesons [5]. The spin symmetry occurs when $\Delta(r) = S(r) - V(r) =$ constant or the scalar potential $S(r)$ is nearly equal to the vector potential $V(r)$, i.e. $S(r) \approx V(r)$. The p-spin symmetry occurs when $\Sigma(r) = S(r) + V(r) =$ constant or $S(r) \approx -V(r)$ [6–9].

Furthermore, the p-spin symmetry refers to a quasi-degeneracy of single-nucleon doublets with non-relativistic quantum number $(n,\ \ell,\ j = \ell + 1/2)$ and $(n-1,\ \ell+2,\ j = \ell + 3/2)$, where n, ℓ and j are single-nucleon radial, orbital and total angular quantum numbers, respectively [10, 11]. The total angular momentum is $j = \tilde{\ell} + \tilde{s}$, where $\tilde{\ell} = \ell + 1$ is pseudo-angular momentum and $\tilde{s}$ is p-spin angular momentum [1]. Recently, the tensor potential was introduced in the Dirac equation with the substitution: $\vec{P} \to \vec{P} - \mathrm{i}m\,\omega\beta.\hat{r}U(r)$ and a spin–orbit coupling is added to the Dirac Hamiltonian [12–17].

The Yukawa potential or static screened Coulomb potential [18–20] can be defined as

$$V(r) = -V_0 \frac{\mathrm{e}^{-ar}}{r}, \tag{1}$$

where $V_0 = \alpha Z$, $\alpha = (137.037)^{-1}$ is the fine-structure constant, Z is the atomic number and a is the screening parameter. This potential is often used to compute bound-state normalizations and energy levels of neutral atoms [21–23]. Over the last few years, several methods have been used in solving relativistic and non-relativistic equations with the Yukawa potential, for example, the shifted large method [24], perturbative solution of the Riccati equation [25, 26], alternative perturbative scheme [27, 28], the quasi-linearization method [29] and the Nikiforov–Uvarov method [30].

The tensor coupling, a higher order term in a relativistic expansion, significantly increases the spin–orbit coupling. This suggests that the tensor coupling could have a significant contribution to p-spin splittings in nuclei as well. This contribution is expected to be particularly relevant for the levels near the Fermi surface, because the tensor coupling depends on the derivative of a vector potential, which has a peak near the Fermi surface for typical nuclear mean-field vector potentials. It has also been used as a natural way to introduce the harmonic oscillator in a relativistic (Dirac) formalism. In a recent paper, it was shown that the harmonic oscillator with scalar and vector potentials can exhibit an exact p-spin symmetry [14, 31]. When this symmetry is broken $(\Sigma \neq 0)$, the breaking term is quite large, manifesting its non-perturbative behavior. However, if a tensor coupling is

introduced, the form of the harmonic-oscillator potential can still be maintained with ($\Sigma = 0$), but the p-spin symmetry is broken perturbatively [32].

The tensor interaction has also been considered to explain how the spin–orbit term can be small for Λ-nucleus and large in the nucleon–nucleus case [33]. It is assumed that in the strange sector (the case of Λ) the tensor coupling is large and the spin–orbit term obtained from this interaction can cancel in part the contribution coming from the scalar and vector interactions. This result shows that the tensor interaction can change strongly the spin–orbit term.

It is therefore the aim of this work to apply the asymptotic iteration method (AIM) [34–37] to solve the Dirac equation with the Yukawa potential including a Coulomb-like tensor interaction to obtain the energy eigenvalues and corresponding wave functions in view of spin and p-spin symmetry.

This paper is organized as follows. In section 2, the asymptotic iteration method (AIM) is briefly introduced. In section 3, we present the Dirac equation with scalar and vector potentials for an arbitrary spin–orbit coupling number including tensor interaction in view of spin and p-spin symmetry. In section 4, we obtain the energy eigenvalue equations and corresponding wave functions. We discuss our numerical results in section 5. The conclusion is given in section 6.

2. Method of analysis

One of the calculational tools used in solving the Schrödinger-like equation including the centrifugal barrier and/or the spin–orbit coupling term is the asymptotic iteration method (AIM). For a given potential the idea is to convert the Schrödinger-like equation to the homogeneous linear second-order differential equation of the form

$$y''(x) = \lambda_o(x)y'(x) + s_o(x)y(x), \tag{2}$$

where $\lambda_o(x)$ and $s_o(x)$ have sufficiently many continuous derivatives and are defined in some intervals which are not necessarily bounded. The differential equation (2) has a general solution [10, 34]

$$y(x) = \exp\left(-\int^x \alpha(x')\,\mathrm{d}x'\right) \times \left[C_2 + C_1 \int^x \exp\left(\int^{x'} \left[\lambda_o(x'') + 2\alpha(x'')\right] \mathrm{d}x''\right) \mathrm{d}x'\right]. \tag{3}$$

If $k > 0$, for sufficiently large k, we obtain the $\alpha(x)$

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \quad k = 1, 2, 3, \ldots, \tag{4}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_o(x)\lambda_{k-1}(x), \\ s_k(x) &= s'_{k-1}(x) + s_o(x)\lambda_{k-1}(x), \quad k = 1, 2, 3, \ldots \end{aligned} \tag{5}$$

with the quantization condition

$$\delta_k(x) = \begin{vmatrix} \lambda_k(x) & s_k(x) \\ \lambda_{k+1}(x) & s_{k+1}(x) \end{vmatrix} = 0, \quad k = 1, 2, 3, \ldots. \tag{6}$$

The energy eigenvalues are then obtained from (6) if the problem is exactly solvable.

3. The Dirac equation with a tensor interaction

In spherical coordinates, the Dirac equation for fermonic massive spin-$\frac{1}{2}$ particles interacting with an arbitrary scalar potential $S(r)$, the time component $V(r)$ of a four-vector potential and the tensor potential $U(r)$ can be expressed as [1, 7]

$$\left[\vec{\alpha}.\vec{p} + \beta(M + S(r)) - \mathrm{i}\,\beta\vec{\alpha}\cdot\hat{r}U(r)\right]\psi(\vec{r}) = [E - V(r)]\psi(\vec{r}), \tag{7}$$

where E, $\vec{p}$ and M denote the relativistic energy of the system, the momentum operator and mass of the particle, respectively. α and β are 4×4 Dirac matrices given by

$$\bar{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -1 \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

where I is the 2×2 unitary matrix and $\vec{\sigma}$ are the three-vector Pauli spin matrices. The eigenvalues of the spin–orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) > 0$ and $\kappa = -\left(j + \frac{1}{2}\right) < 0$ for the unaligned spin $j = \ell - \frac{1}{2}$ and the aligned spin $j = \ell + \frac{1}{2}$, respectively. The set (H^2, K, J^2, J_Z) can be taken as the complete set of conservative quantities with $\vec{J}$ being the total angular momentum operator and $K = (\vec{\sigma}.\vec{L} + 1)$ is the spin–orbit where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa}(\vec{r}) \\ g_{n\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)Y^{\ell}_{jm}(\theta,\phi) \\ \mathrm{i}\,G_{n\kappa}(r)Y^{\tilde{\ell}}_{jm}(\theta,\phi), \end{pmatrix}, \tag{9}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper- and lower-spinor components, respectively, and $Y^{\ell}_{jm}(\theta,\phi)$ and $Y^{\tilde{\ell}}_{jm}(\theta,\phi)$ are the spherical harmonic functions coupled to the total angular momentum j and its projection m on the z-axis. Substitution of equation (7) into equation (2) yields the following coupled differential equations:

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right) F_{n\kappa}(r) &= (M - E_{n\kappa} - \Delta(r))G_{n\kappa}(r), \\ \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right) G_{n\kappa}(r) &= (M - E_{n\kappa} - \Sigma(r))F_{n\kappa}(r), \end{aligned} \tag{10}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and sum potentials, respectively. After eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in equations (10), we obtain the following two Schrödinger-like differential equations for the upper- and lower-spinor components:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - U^2(r) - \frac{\mathrm{d}U(r)}{\mathrm{d}r} + \frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{M + E_{n\kappa} - \Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right)\right] F_{n\kappa}(r) = \left[(M + E_{n\kappa} - \Delta(r))\,(M - E_{n\kappa} + \Sigma(r))\right] F_{n\kappa}(r), \tag{11}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)-U^2(r)+\frac{dU(r)}{\mathrm{d}r}\right.$$
$$\left.+\frac{\frac{d\Sigma(r)}{\mathrm{d}r}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right]G_{n\kappa}(r)$$
$$=[(M+E_{n\kappa}-\Delta(r))\,(M-E_{n\kappa}+\Sigma(r))]\,G_{n\kappa}(r), \quad (12)$$

where $\kappa(\kappa-1)=\bar{\ell}(\bar{\ell}+1)$ and $\kappa(\kappa+1)=\ell(\ell+1)$.

3.1. p-spin symmetry limit

The p-spin symmetry occurs when $\frac{\mathrm{d}[V(r)+S(r)]}{\mathrm{d}r}=\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}=0$ or $\Sigma(r)=C_{\mathrm{ps}}=$ constant. Here we are taking $\Delta(r)$ as the Yukawa potential and the tensor as the Coulomb-like potential, i.e.

$$\Delta(r)=-V_o\frac{\mathrm{e}^{-ar}}{r} \text{ and } U(r)=-\frac{A}{r} \quad r\geqslant R_{\mathrm{c}} \quad (13)$$

with

$$A=\frac{Z_aZ_b\mathrm{e}^2}{4\pi\epsilon_o}, \quad (14)$$

where R_{c} is the Coulomb radius, and Z_a and Z_b, respectively, denote the charges of the projectile a and the target nuclei b [1, 7]. Under this symmetry, equation (12) can easily be transformed to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\tilde{\delta}}{r^2}+\frac{\tilde{\gamma}\,\mathrm{e}^{-ar}}{r}-\tilde{\beta}^2\right]G_{n\kappa}(r)=0, \quad (15)$$

where $\kappa=-\tilde{\ell}$ and $\kappa=\tilde{\ell}+1$ for $\kappa<0$ and $\kappa>0$, respectively, and

$$\tilde{\gamma}=(E_{n\kappa}-M-C_{\mathrm{ps}})V_0,$$
$$\tilde{\beta}=\sqrt{(M+E_{n\kappa})(M-E_{n\kappa}+C_{\mathrm{ps}})}, \quad (16)$$
$$\tilde{\delta}=(\kappa+A)(\kappa+A-1)$$

have been introduced for simplicity.

3.2. Spin symmetry limit

In the spin symmetry limit, $\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}=0$ or $\Delta(r)=C_{\mathrm{s}}=$ constant [1, 7]. Similarly to section 3.1, we consider

$$\Sigma(r)=-V_o\frac{\mathrm{e}^{-ar}}{r} \quad \text{and} \quad U(r)=-\frac{A}{r} \quad r\geqslant R_{\mathrm{c}}. \quad (17)$$

With equations (17), equation (11) can be transformed to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\delta}{r^2}+\frac{\gamma\,\mathrm{e}^{-ar}}{r}-\beta^2\right]F_{n\kappa}(r)=0, \quad (18)$$

where $\kappa=\ell$ and $-\ell-1$ for $\kappa<0$ and $\kappa>0$, respectively. We have also introduced the following parameters:

$$\gamma=(M+E_{n\kappa}-C_{\mathrm{s}})V_0,$$
$$\beta=\sqrt{(M-E_{n\kappa})(M+E_{n\kappa}-C_{\mathrm{s}})}, \quad (19)$$
$$\delta=(\kappa+A)(\kappa+A+1)$$

for simplicity.

4. Approximate relativistic bound states

In this section, within the framework of the AIM, we shall solve the Dirac equation with the Yukawa potential in the presence of the tensor potential.

4.1. Pseudospin symmetric solution

To obtain the analytical approximate solution for the Yukawa potential, an approximation has to be made for the centrifugal term $1/r^2$, which is similar to the one taken by other authors [37–43]

$$\frac{1}{r^2}\approx\frac{4a^2\,\mathrm{e}^{-2ar}}{\left(1-\mathrm{e}^{-2ar}\right)^2}, \quad (20)$$

which is valid for $ar\ll 1$. With this approximation, equation (15) can be written as

$$\frac{\mathrm{d}^2G_{n\kappa}(r)}{\mathrm{d}r^2}+\left[\frac{2a\tilde{\gamma}\mathrm{e}^{-2ar}}{\left(1-\mathrm{e}^{-2ar}\right)}-\frac{4a^2\tilde{\delta}\mathrm{e}^{-2ar}}{\left(1-\mathrm{e}^{-2ar}\right)^2}-\tilde{\beta}^2\right]G_{n\kappa}(r)=0. \quad (21)$$

In order to obtain the solution of equation (21), we introduce a transformation of the form $z=\left(\mathrm{e}^{-2ar}-1\right)^{-1}$; as a result, equation (21) can be rewritten as

$$z(z+1)\frac{\mathrm{d}^2G_{n\kappa}(z)}{\mathrm{d}z^2}+(1+2z)\frac{\mathrm{d}G_{n\kappa}(z)}{\mathrm{d}z}$$
$$-\frac{\left(\tilde{\delta}z^2+\left(\tilde{\delta}+\frac{\tilde{\gamma}}{2a}\right)z+\left(\frac{\tilde{\gamma}}{2a}+\frac{\tilde{\beta}^2}{4a^2}\right)\right)}{z(1+z)}G_{n\kappa}(r). \quad (22)$$

According to the Frebenius theorem, the singularity points of the above differential equation play an essential role in the form of the wave functions. The singular points of the above equation (22) are at $z=0$ and -1. As a result, we take the wave functions of the form

$$G_{n\kappa}(z)=z^{\tilde{\rho}}(1+z)^{-\frac{\tilde{\beta}}{2a}}g_{n\kappa}(z), \quad (23)$$

where

$$\tilde{\rho}=\sqrt{\frac{\tilde{\beta}^2}{4a^2}+\frac{\tilde{\gamma}}{2a}}. \quad (24)$$

Substituting equations (23) and (24) into equation (22) allows us to find the following second-order equation:

$$\frac{\mathrm{d}^2g_{n\kappa}(z)}{\mathrm{d}z^2}-\left[\frac{2z\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}-1\right)-(2\tilde{\rho}+1)}{z(1+z)}\right]\frac{\mathrm{d}g_{n\kappa}}{\mathrm{d}z}$$
$$-\left[\frac{\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}\right)-\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}\right)^2+\tilde{\delta}}{z(z+1)}\right]g_{n\kappa}(z), \quad (25)$$

which is suitable for an AIM solution. In order to use the AIM procedure, we compare equation (25) with equation (2) and obtain the $\lambda_0(z)$ and $s_0(z)$ equations as

$$\lambda_0(z)=\frac{2z\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}-1\right)-(2\tilde{\rho}+1)}{z(1+z)},$$
$$s_0(z)=\frac{\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}\right)-\left(\frac{\tilde{\beta}}{2a}-\tilde{\rho}\right)^2+\tilde{\delta}}{z(z+1)}. \quad (26)$$

By using the termination condition of the AIM given in equation (6), we obtain

$$\delta_0(z)=\begin{vmatrix}\lambda_1(z) & s_1(z)\\ \lambda_0(z) & s_o(z)\end{vmatrix}=0 \Rightarrow \tilde{\rho}_0-\frac{\tilde{\beta}_0}{2a}=-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\tilde{\delta}},$$
$$\delta_1(z)=\begin{vmatrix}\lambda_2(z) & s_2(z)\\ \lambda_1(z) & s_1(z)\end{vmatrix}=0 \Rightarrow \tilde{\rho}_1-\frac{\tilde{\beta}_1}{2a}=-\frac{3}{2}-\frac{1}{2}\sqrt{1+4\tilde{\delta}},$$
$$\delta_2(z)=\begin{vmatrix}\lambda_3(z) & s_3(z)\\ \lambda_2(z) & s_2(z)\end{vmatrix}=0 \Rightarrow \tilde{\rho}_2-\frac{\tilde{\beta}_2}{2a}=-\frac{5}{2}-\frac{1}{2}\sqrt{1+4\tilde{\delta}},$$
$$\ldots\text{etc.} \tag{27}$$

The above expressions can be generalized as

$$\tilde{\rho}_n-\frac{\tilde{\beta}_n}{2a}=-\frac{2n+1}{2}-\frac{1}{2}\sqrt{1+4\tilde{\delta}}. \tag{28}$$

If one inserts the values of $\tilde{\rho}$, $\tilde{\beta}$ and $\tilde{\delta}$ into equation (28), the energy spectrum equation can be obtained as

$$\sqrt{(M+E_{n\kappa})\left(M-E_{n\kappa}+C_{\rm ps}\right)}-2a(\kappa+A+n)$$
$$=\sqrt{\left(M-E_{n\kappa}+C_{\rm ps}\right)(M+E_{n\kappa}-2aV_o)}. \tag{29}$$

On squaring both sides of equation (29), we obtain a more explicit expression for the energy spectrum as

$$\frac{\left[2a(n+\kappa+A)^2+V_o(M-E_{n\kappa}+C_{\rm ps})\right]^2}{4(M+E_{n\kappa})(M-E_{n\kappa}+C_{\rm ps})}=(n+\kappa+A)^2. \tag{30}$$

For the special case when $A=0$, our result is exactly identical with the one obtained by Ikhdair [41] and Aydoğdu and Sever [42]. Now we shall obtain the eigenfunction using the AIM. Generally speaking, the differential equation we wish to solve should be transformed to the form [35]

$$y''(x)=2\left(\frac{\Lambda x^{N+1}}{1-bx^{N+2}}-\frac{m+1}{x}\right)y'(x)-\frac{Wx^N}{1-bx^{N+2}}, \tag{31}$$

where a, b and m are constants. The general solution of equation (31) is found to be [2]

$$y_n(x)=(-1)^nC_2(N+2)^n(\sigma)_n{}_2F_1(-n,t+n;\sigma;bx^{N+2}), \tag{32}$$

where the following notations have been used:

$$(\sigma)_n=\frac{\Gamma(\sigma+n)}{\Gamma(\sigma)},\quad \sigma=\frac{2m+N+3}{N+2}$$
$$\text{and}\quad t=\frac{(2m+1)b+2\Lambda}{(N+2)b}. \tag{33}$$

Now, comparing equations (32) and (22), we have $\Lambda=\frac{\tilde{\beta}}{2a}-\frac{1}{2}$, $b=-1$, $N=-1$, $m=\tilde{\rho}-\frac{1}{2}$, $\sigma=2\tilde{\rho}+1$, $t=2\left(\tilde{\rho}-\frac{\tilde{\beta}}{2a}\right)$ and then the solution of equation (25) can easily be found as

$$g_{n\kappa}(z)=(-1)^n\frac{\Gamma(2\tilde{\rho}+1)}{\Gamma(2\tilde{\rho})}{}_2F_1$$
$$\times\left(-n,2\left(\tilde{\rho}-\frac{\tilde{\beta}}{2a}\right)+n+1;2\tilde{\rho}+1;-z\right), \tag{34}$$

where Γ and ${}_2F_1$ are the Gamma function and hypergeometric function, respectively. Using equations (23) and (34), we can write the corresponding lower spinor component $G_{n\kappa}(z)$ as

$$G_{n\kappa}(z)=N_{n\kappa}z^{\sqrt{\frac{\tilde{\beta}^2}{4a^2}+\frac{\tilde{\gamma}}{2a}}}(1+z)^{-\frac{\tilde{\beta}}{2a}}{}_2F_1$$
$$\times\left(-n,2\left(\sqrt{\frac{\tilde{\beta}^2}{4a^2}+\frac{\tilde{\gamma}}{2a}}-\frac{\tilde{\beta}}{2a}\right)+n+1;\right.$$
$$\left.2\sqrt{\frac{\tilde{\beta}^2}{4a^2}+\frac{\tilde{\gamma}}{2a}}+1;-z\right), \tag{35}$$

where $N_{n\kappa}$ is the normalization constant.

4.2. Spin symmetric solution

Following section 4.1, approximation (20) is used instead of the centrifugal term $1/r^2$ and we rewrite equation (18) as

$$\frac{\mathrm{d}^2F_{n\kappa}(r)}{\mathrm{d}r^2}+\left[\frac{2a\gamma\mathrm{e}^{-2ar}}{\left(1-\mathrm{e}^{-2ar}\right)}-\frac{4a^2\delta\mathrm{e}^{-2ar}}{\left(1-\mathrm{e}^{-2ar}\right)^2}-\beta^2\right]F_{n\kappa}(r)=0. \tag{36}$$

We have decided to use the same variable so as to avoid repetition of algebra. It is clear that equation (36) is similar to equation (21); therefore, substituting for γ, δ and β in equation (28), the relativistic energy spectrum turns out to be

$$\frac{\left[2a(n+\kappa+A+1)^2-V_0(M+E_{n\kappa}-C_{\rm s})\right]^2}{4(M-E_{n\kappa})(M+E_{n\kappa}-C_{\rm s})}=(n+\kappa+A+1)^2 \tag{37}$$

and the associated upper spinor component $F_{n\kappa}(z)$ is

$$F_{n,\kappa}(z)=C_{n,\kappa}z^{\sqrt{\frac{\beta^2}{4a^2}+\frac{\gamma}{2a}}}(1+z)^{-\frac{\beta}{2a}}{}_2F_1$$
$$\times\left(-n,2\left(\sqrt{\frac{\beta^2}{4a^2}+\frac{\gamma}{2a}}-\frac{\beta}{2a}\right)+n+1;\right.$$
$$\left.2\sqrt{\frac{\beta^2}{4a^2}+\frac{\gamma}{2a}}+1;-z\right), \tag{38}$$

where $C_{n,\kappa}$ is the normalization constant.

Unlike the non-relativistic case, the normalization condition for the Dirac spinor combines the two individual normalization constants $N_{n,\kappa}$ and $C_{n,\kappa}$ in a single integral. The radial wave functions are normalized according to the formula $\int_0^\infty\psi^\dagger\psi r^2\,\mathrm{d}r=1$, which explicitly implies for $f_{n,\kappa}(r)$ and $g_{n,\kappa}(r)$ in equation (9) that [44]

$$\int_0^\infty\left(f_{n,\kappa}^2(r)+g_{n,\kappa}^2(r)\right)r^2\,\mathrm{d}r$$
$$=\int_0^\infty\left(F_{n,\kappa}^2(r)+G_{n,\kappa}^2(r)\right)\mathrm{d}r=1. \tag{39}$$

where the upper and lower spinor components of the total radial wave functions can be expressed in terms of the confluent hypergeometric functions as [41]

$$F_{n\kappa}(s)=N\frac{\Gamma(n+2\beta+1)}{\Gamma(2\beta+1)n!}s^\beta(1-s)^{\kappa+A+1}$$
$$\times{}_2F_1\left(-n,2(\beta+\kappa+A+1)+n;2\beta+1;s\right) \tag{40}$$

and

$$G_{n\kappa}(s) = N\frac{\Gamma(n+2\gamma+1)}{\Gamma(2\gamma+1)n!}s^{\gamma}(1-s)^{\kappa+A} \times {}_2F_1\left(-n, 2(\gamma+\kappa+A+1)+n; 2\gamma+1; s\right) \quad (41)$$

with

$$\beta = \sqrt{(M-E_{n\kappa})(M+E_{n\kappa}-C_s)},$$
$$\gamma = \sqrt{(M+E_{n\kappa})(M-E_{n\kappa}+C_{ps})}, \quad s = e^{-2ar}. \quad (42)$$

Hence, equation (39) can be re-written in terms of variable s as

$$\int_0^1 \frac{ds}{s}\left[F_{n,\kappa}^2(s)+G_{n\kappa}^2(s)\right] = 2a, \quad (43)$$

where $s \to 1$ when $r \to 0$ and $s \to 0$ when $r \to \infty$. The method to compute N is given in [45]. Thus we have

$$N^2\left[\left(\frac{\Gamma(n+2\beta+1)}{\Gamma(2\beta+1)n!}\right)^2\int_0^1 s^{2\beta-1}(1-s)^{2(\kappa+A+1)} \times \left[{}_2F_1\left(-n, 2(\beta+\kappa+A+1)+n; 2\beta+1; s\right)\right]^2 ds\right]$$
$$+N^2\left[\left(\frac{\Gamma(n+2\gamma+1)}{\Gamma(2\gamma+1)n!}\right)^2\int_0^1 s^{2\gamma-1}(1-s)^{2(\kappa+A)} \times \left[{}_2F_1\left(-n, 2(\gamma+\kappa+A)+n; 2\gamma+1; s\right)\right]^2 ds\right] = 2a \quad (44)$$

with the confluent hypergeometric function which is defined by

$${}_pF_q\left(a_1,\ldots a_p; b_1,\ldots b_q; s\right) = \sum_{i=0}^{\infty}\frac{(a_1)_i\ldots(a_p)_i s^i}{(b_1)_i\ldots(b_q)_i i!}, \quad (45)$$

where $(a_1)_i$ and $(b_1)_i$ are Pochhammer symbols. We can obtain the normalization constant as [46]

$$N^2\left\{\left(\frac{\Gamma(n+2\beta+1)}{\Gamma(2\beta+1)n!}\right)^2\sum_{i=0}^{\infty}\frac{(-n)_i(2\beta+2\kappa+2A+2+n)_i(2\beta)_i}{(2\beta+1)_i(2\beta+2\kappa+2A+3)_i i!} \times B(2\beta, 2\kappa+2A+3)\ {}_3F_2(-n, 2(\beta+\kappa+A+1)+n, \times 2\beta+i; 2\beta+1, 2(\beta+\kappa+A+1)+1+i; 1)\right.$$
$$+\left(\frac{\Gamma(n+2\gamma+1)}{\Gamma(2\gamma+1)n!}\right)^2\sum_{i=0}^{\infty}\frac{(-n)_i(2\gamma+2\kappa+2A+n)_i(2\gamma)_i}{(2\gamma+1)_i(2\gamma+2\kappa+2A)_i i!} \times B(2\gamma, 2\kappa+2A+1)\ {}_3F_2(-n, 2(\gamma+\kappa+A)+n, 2\gamma+i; \times 2\gamma+1, 2(\gamma+\kappa+A)+1+i; 1)\Bigg\} = 2a, \quad (46)$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \mathbf{Re}(x),\ \mathbf{Re}(y) > 0,$$
$$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi,\ B(x,y) = B(y,x). \quad (47)$$

The incomplete beta function is given as

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt \quad (48)$$

and the regularized beta function as

$$I_x(a,b) = \frac{B(x;a,b)}{B(a,b)}. \quad (49)$$

In the special case when $A = 0$, our result is identical to the one obtained by Ikhdair [41] and also by Setare and Haidari [43] by means of the Nikiforov–Uvarov method.

5. Numerical results

By taking $V_0 = 1$ and $C_{ps} = -5.0\,\text{fm}^{-1}$, we found that the particle is strongly attracted to the nucleus. In the absence of tensor interaction, i.e. $A = 0$, we noted that the set of p-spin symmetry doublets: $(1s_{1/2}, 2p_{3/2})$, $(1p_{3/2}, 2s_{1/2}, 2d_{5/2})$, $(1d_{5/2}, 2f_{7/2})$, $(0d_{3/2}, 1f_{7/2})$, $(0f_{5/2}, 1d_{3/2})$, $(0g_{7/2}, 1f_{5/2})$ and $(0h_{9/2}, 1g_{7/2})$ have same energies as $-4.500\,000\,000$, $-4.497\,559\,159$, $-4.490\,219\,111$, $-4.477\,926\,642$, $-4.460\,590\,892$, $-4.438\,679\,893$ and $-4.410\,215\,299\,\text{fm}^{-1}$, respectively.

It is also noted that the presence of tensor interaction, say $A = 0.5$, removes the degeneracy between the states in the above doublets and creates a new set of p-spin symmetric doublets $(1s_{1/2}, 1p_{3/2}, 2p_{3/2}, 2d_{5/2})$, $(1d_{5/2}, 2f_{7/2}, 2s_{1/2})$, $(1f_{7/2})$, $(0d_{3/2})$, $(0f_{5/2}, 1d_{3/2})$, $(0g_{7/2}, 1f_{5/2})$ and $(0h_{9/2}, 1g_{7/2})$ having identical energies as $-4.499\,390\,062$, $-4.494\,504\,009$, $-4.484\,696\,693$, $-4.469\,896\,410$, $-4.449\,992\,282$, $-4.424\,829\,942$ and $-4.394\,205\,177\,\text{fm}^{-1}$, respectively.

With the strength of tensor interaction increasing, say $A = 1.0$, the following p-spin doublets have the same energies as the counter ones when $A = 0$ as $(1p_{3/2}, 2d_{5/2}) \leftrightarrow (1s_{1/2}, 2p_{3/2})$, $(1s_{1/2}, 1d_{5/2}, 2p_{3/2}, 2f_{7/2}) \leftrightarrow (1p_{3/2}, 2s_{1/2}, 2d_{5/2})$, $(1f_{7/2}, 2s_{1/2}) \leftrightarrow (1d_{5/2}, 2f_{7/2})$, $(0d_{3/2}) \leftrightarrow (0f_{5/2}, 1d_{3/2})$, $(0f_{5/2}, 1d_{3/2}) \leftrightarrow (0g_{7/2}, 1f_{5/2})$ and $(0g_{7/2}, 1f_{5/2}) \leftrightarrow (0h_{9/2}, 1g_{7/2})$.

Furthermore, we considered the case where $C_{ps} = 0$ and the same set of p-spin symmetry doublets: $(1s_{1/2}, 2p_{3/2})$, $(1p_{3/2}, 2s_{1/2}, 2d_{5/2})$, $(1d_{5/2}, 2f_{7/2})$, $(0d_{3/2}, 1f_{7/2})$, $(0f_{5/2}, 1d_{3/2})$, $(0g_{7/2}, 1f_{5/2})$ and $(0h_{9/2}, 1g_{7/2})$ have the same energies as $-4.500\,000\,000$, $-0.208\,711\,915$, $-0.300\,000\,000$, $-0.260\,536\,466$, $-0.140\,000\,000$, $-2.823\,651\,852$ and $-5.111\,034\,483\,\text{fm}^{-1}$, respectively, whereas the doublet set $(1s_{1/2}, 2p_{3/2})$ has no bound negative energy.

When the tensor strength $A = 0.5$, the degeneracy is changed as $0d_{3/2}$, $(1d_{5/2}, 2s_{1/2}, 2f_{7/2})$, $1f_{7/2}$, $(0f_{5/2}, 1d_{3/2})$, $(0g_{7/2}, 1f_{5/2})$ and $(0h_{9/2}, 1g_{7/2})$ have the bound energies $-0.213\,820\,459$, $-0.284\,176\,740$, $-0.289\,224\,031\,2$, $-0.586\,585\,366$, $-4.074\,227\,224$ and $-6.043\,369\,573$, respectively, whereas the set $(1s_{1/2}, 1p_{3/2}, 2d_{5/2}, 2p_{3/2})$ has no negative bound energy. The particle becomes strongly bounded as n and κ increases.

By increasing the tensor strength as $A = 1$, the following p-spin doublets have same energies as in the case $A = 0$: $0d_{3/2} \to -0.140\,000\,000\,\text{fm}^{-1}$,

Table 1. The p-spin symmetric energy eigenvalues of the Yukawa potential for various values of n and κ with $M = 0.5$, $V_0 = 1.0$ and $a = 0.1\,\mathrm{fm}^{-1}$ for various C_{ps} values.

	$E_{n\kappa}(\mathrm{fm}^{-1})$				
C_{ps}	$1s_{1/2}$	$1p_{3/2}$	$2d_{5/2}$	$0f_{5/2}$	$1g_{7/2}$
−40	−39.499 744 24	−39.500 000 000	−39.499 744 24	−39.500 000 000	−39.487 463 99
	−8.220 255 757	–	−8.220 255 757	–	−0.611 013 166
−35	−34.499 706 74	−34.500 000 000	−34.499 706 74	−34.500 000 000	−34.485 624 41
	−7.220 293 258	–	−7.220 293 258	–	−0.587 472 039
−30	−29.499 656 35	−29.500 000 000	−29.499 656 35	−29.500 000 000	−29.483 151 71
	−6.220 343 648	–	−6.220 343 648	–	−0.564 564 029
−25	−24.499 585 05	−24.500 000 000	−24.499 585 05	−24.500 000 000	−24.479 650 78
	−5.220 414 947	–	−5.220 414 947	–	−0.542 684 245
−20	−19.499 476 42	−19.500 000 000	−19.499 476 42	−19.500 000 000	−19.474 310 82
	−4.220 523 578	–	−4.220 523 578	–	−0.522 643 493
−15	−14.499 290 74	−14.500 000 000	−14.499 290 74	−14.500 000 000	−14.465 161 71
	−3.220 709 264	–	−3.220 709 264	–	−0.506 411 895
−10	−9.498 900 933	−9.500 000 000	−9.498 900 933	−9.500 000 000	−9.445 829 737
	−2.221 099 067	–	−2.221 099 067	–	−0.500 363 156
−5	−4.497 559 159	−4.500 000 000	−4.497 559 159	−4.500 000 000	−4.376 764 780
	−1.222 440 841	–	−1.222 440 841	–	−0.544 047 403

Table 2. The spin symmetric energy eigenvalues of the Yukawa potential for various values of n and κ with $M = 0.5$ $V_0 = 1.0$ and $a = 0.1\,\mathrm{fm}^{-1}$ for various C_{s} values.

	$E_{n\kappa}(\mathrm{fm}^{-1})$				
C_{s}	$1s_{1/2}$	$1p_{3/2}$	$0p_{1/2}$	$2f_{7/2}$	$1f_{5/2}$
5	4.489 714 770	4.497 433 787	4.476 781 001	4.500 000 000	4.405 380 777
	0.839 696 995	1.382 566 213	0.728 624 405	–	0.721 515 775
10	9.495 503 204	9.498 876 227	9.489 875 804	9.500 000 000	9.459 363 733
	1.128 026 208	2.381 123 773	0.850 664 737	–	0.702 015 577
15	14.497 121 670	14.499 280 530	14.493 522 080	14.500 000 000	14.474 051 94
	1.420 525 390	3.380 719 471	0.982 153 599	–	0.721 810 125
20	19.497 883 350	19.499 470 880	19.495 236 860	19.500 000 000	19.480 933 010
	1.713 881 360	4.380 529 119	1.115 573 949	–	0.749 411 816
25	24.498 326 240	24.499 581 580	24.500 000 000	24.500 000 000	24.484 927 67
	2.007 556 118	5.380 418 419	1.249 712 246	–	0.779 899 919
30	29.498 615 850	29.499 653 970	29.496 885 470	29.500 000 000	29.487 537 840
	2.301 384 153	6.380 346 026	1.384 195 613	–	0.811 772 503
35	34.498 820 020	34.499 705 010	34.497 344 920	34.500 000 000	34.489 377 180
	2.595 297 632	7.380 294 988	1.518 871 297	–	0.844 415 924
40	39.498 971 690	39.499 742 930	39.497 686 230	39.500 000 000	39.490 743 280
	2.889 263 601	8.380 257 072	1.653 665 117	–	0.877 532 579

$(1s_{1/2}, 1d_{5/2}, 2p_{3/2}, 2f_{7/2}) \leftrightarrow (1p_{3/2}, 2s_{1/2}, 2d_{5/2})$, $(1f_{7/2}, 2s_{1/2}) \leftrightarrow (1d_{5/2}, 2f_{7/2})$, $(0f_{5/2}, 1d_{3/2}) \leftrightarrow (0g_{7/2}, 1f_{5/2})$ and $(0g_{7/2}, 1f_{5/2}) \leftrightarrow (0h_{9/2}, 1g_{7/2})$. The p-spin doublet set $(1p_{3/2}, 2d_{5/2})$ has no negative bound state energy.

Let us also discuss the energy states in the presence of spin symmetry for $C_{\mathrm{s}} = 5.0$ and $0\,\mathrm{fm}^{-1}$, respectively. In the absence of tensor interaction and when $C_{\mathrm{s}} = 5.0\,\mathrm{fm}^{-1}$, the degeneracy in the sets $(0s_{1/2}, 1p_{3/2}, 2d_{5/2}, 3f_{7/2})$, $(1s_{1/2}, 0p_{3/2}, 2p_{3/2}, 1d_{5/2}, 3d_{5/2}, 2f_{7/2})$, $(2f_{5/2}, 3d_{3/2})$ $(2s_{1/2}, 2p_{3/2}, 0d_{5/2}, 3d_{5/2}, 1f_{7/2}, 0p_{1/2})$, $(3s_{1/2}, 1p_{1/2}, 0f_{7/2}, 0d_{3/2})$, $(2p_{1/2}, 0f_{5/2}, 1d_{3/2})$ and $(3p_{1/2}, 1f_{5/2}, 2d_{3/2})$ is shown.

Furthermore, in the presence of tensor interaction $A = 1$, the following doublet sets have identical energies as their counter ones in the absence of a tensor: $(0p_{3/2}, 1d_{5/2}, 2f_{7/2}) \leftrightarrow (0s_{1/2}, 1p_{3/2}, 2d_{5/2}, 3f_{7/2})$, $(0s_{1/2}, 1p_{3/2}, 0d_{5/2}, 2d_{5/2}, 1f_{7/2}, 3f_{7/2}) \leftrightarrow (1s_{1/2}, 0p_{3/2}, 2p_{3/2}, 1d_{5/2}, 3d_{5/2}, 2f_{7/2})$, $(1s_{1/2}, 2p_{3/2}, 3d_{5/2}, 0f_{7/2}) \leftrightarrow (2s_{1/2}, 2p_{3/2}, 0d_{5/2}, 3d_{5/2}, 1f_{7/2}, 0p_{1/2})$, $(2s_{1/2}, 3p_{3/2}, 0p_{1/2}) \leftrightarrow (3s_{1/2}, 1p_{1/2}, 0f_{7/2}, 0d_{3/2})$, $(3s_{1/2}, 1p_{1/2}, 0d_{3/2}) \leftrightarrow (2p_{1/2}, 0f_{5/2}, 1d_{3/2})$, $(2p_{1/2}, 1d_{3/2}, 0f_{5/2}) \leftrightarrow (3p_{1/2}, 1f_{5/2}, 2d_{3/2})$ and $(3p_{1/2}, 2d_{3/2}, 1f_{5/2}) \leftrightarrow (2f_{5/2}, 3d_{3/2})$.

In table 1, in the absence of tensor interaction and when C_{ps} becomes more negative, the system is becoming strongly attractive with the p-spin doublets $(1s_{1/2}, 2d_{5/2})$ and $(1p_{3/2}, 0f_{5/2})$ are degenerate sets for all values of C_{ps}. In the presence of spin symmetry, the increasing of C_{s} value forces the system to become more repulsive and the considered states $1s_{1/2}$, $2p_{3/2}$, $0p_{1/2}$, $2f_{7/2}$ and $1f_{5/2}$ are not degenerate for all values of C_{s} as shown in table 2.

6. Concluding remarks

In this paper, we have obtained the approximate bound states of a Dirac particle confined to the field of the Yukawa potential and the tensor Coulomb-type interaction in the form of $-A/r$. We used the AIM to obtain the energy eigenvalues and wave functions in closed form in the presence of the spin and the p-spin symmetries. In the presence of p-spin symmetry, some numerical values of the energy levels are calculated in table 1 for various values of $C_{\mathrm{ps}} = -5$ to $-40\,\mathrm{fm}^{-1}$. Also, in the presence of spin symmetry, the numerical energy levels are calculated in table 2 for

various values of $C_s = 5$–$0\,\mathrm{fm}^{-1}$. Obviously, the degeneracy between the members of spin doublets and p-spin doublets is removed by the tensor interaction. The spin and p-spin spectra of the present potential are identical to those obtained in previous works [19, 41–43]. We should remark that the present approximation is valid only for the lowest orbital states [47–49]. Finally, the relativistic spin symmetry in the absence of tensor interaction $-A/r$ and when $C_s = 0\,\mathrm{fm}^{-1}$ can be reduced to the non-relativistic solution for the Yukawa potential.

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References

[1] Ginocchio J N 2005 *Phys. Rep.* **414** 165
[2] Bohr A, Hamamoto I and Mottelson B R 1982 *Phys. Scr.* **26** 267
[3] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 *Phys. Rev. Lett.* **59** 1405
[4] Troltenier D, Bahri C and Draayer J P 1995 *Nucl. Phys.* A **586** 53
[5] Page P R, Goldman T and Ginocchio J N 2001 *Phys. Rev. Lett.* **86** 204
[6] Ginocchio J N, Leviatan A, Meng J and Zhou S G 2004 *Phys. Rev.* C **69** 034303
[7] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436
[8] Oyewumi K J and Akoshile C O 2010 *Eur. Phys. J.* A **45** 311
[9] Oyewumi K J 2012 *Approximate Solutions of the Dirac Equation for the Rosen Morse Potential in the Presence of the Spin–Orbit and Pseudo-Orbit Centrifugal Terms* (*Theoretical Concepts of Quantum Mechanics* vol 445) ed M R Pahlavani (Croatia: INTECH) chapter 19, pp 445–69
[10] Hecht K T and Adler A 1969 *Nucl. Phys.* A **137** 129
[11] Arima A, Harvey M and Shimizu K 1969 *Phys. Lett.* B **30** 517
[12] Ikhdair S M and Sever R 2010 *Appl. Math. Comput.* **216** 911
[13] Moshinsky M and Szczepanika A 1989 *J. Phys. A: Math. Gen.* **22** L817
[14] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev.* C **69** 024319
[15] Alberto P, Lisboa R, Malheiro M and de Castro A S 2005 *Phys. Rev.* C **71** 034313
[16] Akcay H 2009 *Phys. Lett.* A **373** 616
[17] Aydoğdu O and Sever R 2010 *Eur. Phys. J.* A **43** 73
[18] Yukawa H 1935 *Proc. Phys. Math. Soc. Japan* **17** 48
[19] Maghsoodi E, Hassanabadi H and Aydoğdu O 2012 *Phys. Scr.* **86** 015005
[20] Hamzavi M, Movahedi M, Thylwe K-E and Rajabi A A 2012 *Chin. Phys. Lett.* **29** 080302
[21] McEnnan J, Kissel L and Pratt R H 1976 *Phys. Rev.* A **13** 532
[22] Mehta C H and Patil S H 1978 *Phys. Rev.* A **17** 34
[23] Dutt R and Varshni Y P 1983 *Z. Phys.* A **313** 143
Dutt R and Varshni Y P 1986 *Z. Phys.* D **2** 207
Lai C S and Madan M P 1984 *Z. Phys.* A **316** 131
[24] Imbo T, Pagnamenta A and Sukhatme U 1984 *Phys. Lett.* A **105** 183
[25] Chakrabarti B and Das T K 2001 *Phys Lett.* A **285** 11
[26] Gonul B, Koksal K and Bakir E 2006 *Phys. Scr.* **73** 279
[27] Ikhdair S M and Sever R 2006 *Int. J. Mod. Phys.* A **21** 6465
[28] Ikhdair S M and Sever R 2007 *J. Mol. Struct.* **809** 103
[29] Liverts E Z, Drukarev E G, Krivec R and Mandelzweig V B 2008 *Few-Body Syst.* **44** 367
[30] Hamzavi M, Movahedi M, Thylwe K-E and Rajabi A A 2012 *Chin. Phys. Lett.* **29** 080302
[31] Ginocchio J N 2004 *Phys. Rev.* C **69** 034318
[32] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais F 2004 *Int. J. Mod. Phys.* D **13** 1447
[33] Chiapparini M, Gatone A O and Jennings B K 1991 *Nucl. Phys.* A **529** 589
[34] Ciftci H, Hall R L and Saad N 2003 *J. Phys. A: Math. Gen.* **36** 11807
[35] Ciftci H, Hall R L and Saad N 2005 *Phys. Lett.* A **340** 388
[36] Falaye B J 2012 *J. Math. Phys.* **53** 082107
Falaye B J 2012 *Few-Body Syst.* **53** 557
Falaye B J 2012 *Few-Body Syst.* **53** 563
[37] Falaye B J 2012 *Cent. Eur. J. Phys.* **10** 960
[38] Solyu A, Bayrak O and Boztosun I 2008 *J. Phys. A: Math. Theor.* **41** 065308
[39] Hamzavi M, Ikhdair S M and Ita B I 2012 *Phys. Scr.* **85** 045009
[40] Zhang L H, Li X P and Jia C S 2008 *Phys. Lett.* A **372** 2201
[41] Ikhdair S M 2012 *Cent. Eur. J. Phys.* **10** 361
[42] Aydoğdu O and Sever R 2011 *Phys. Scr.* **84** 025005
[43] Setare M R and Haidari S 2010 *Phys. Scr.* **81** 065201
[44] Greiner W 2000 *Relativistic Quantum Mechanics* 3rd edn (Berlin: Springer) p 231
[45] Saad N 2007 *Phys. Scr.* **76** 623
[46] Arfken G B and Weber H J 1995 *Mathematical Methods for Physicists* (San Diego, CA: Academic)
[47] Ikhdair S M 2009 *Eur. Phys. J.* A **39** 307
[48] Ikhdair S M and Sever R 2011 *J. Phys. A: Math. Theor.* **44** 355301
[49] Ikhdair S M, Berkdemir C and Sever R 2011 *Appl. Math. Comput.* **217** 9019