

## ANY $l$ -STATE SOLUTIONS OF THE WOODS–SAXON POTENTIAL IN ARBITRARY DIMENSIONS WITHIN THE NEW IMPROVED QUANTIZATION RULE

SAMEER M. IKHDAIR

*Physics Department, Near East University,  
Nicosia, North Cyprus, Turkey  
sikhdair@neu.edu.tr*

RAMAZAN SEVER

*Physics Department, Middle East Technical University,  
06800, Ankara, Turkey  
sever@metu.edu.tr*

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The approximated energy eigenvalues and the corresponding eigenfunctions of the spherical Woods–Saxon effective potential in  $D$  dimensions are obtained within the new improved quantization rule for all  $l$ -states. The Pekeris approximation is used to deal with the centrifugal term in the effective Woods–Saxon potential. The interdimensional degeneracies for various orbital quantum number  $l$  and dimensional space  $D$  are studied. The solutions for the Hulthén potential, the three-dimensional ( $D = 3$ ), the  $s$ -wave ( $l = 0$ ) and the cases are briefly discussed.

*Keywords:* Woods–Saxon potential; improved quantization rule; Pekeris approximation.

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### 1. Introduction

In recent years, the analytic bound state solution to the hyperradial Schrödinger equation, in any arbitrary spatial dimension ( $D \geq 2$ ), with several potential models has received increasing attention in the literature.<sup>1</sup> There are only a few potentials for which the radial Schrödinger equation can be solved explicitly for all  $n$  and  $l$  quantum numbers. One of these exactly solvable potentials is the Woods–Saxon (WS) potential<sup>2</sup> which has been firstly solved within the Schrödinger equation for  $s$ -states ( $l = 0$ ) by Bose.<sup>3,4</sup> However, the three-dimensional radial Schrödinger equation for the spherical WS potential cannot be solved analytically for  $l \neq 0$  states because of the centrifugal term  $\sim r^{-2}$ . This potential was used as a major part of nuclear shell model to obtain the nuclear energy level spacing and properties of electron distributions in atoms, nuclei and atomic clusters.<sup>5–7</sup> It was used to

describe the interaction of a nucleon (neutron) with a heavy nucleus<sup>8</sup> and also for optical potential model in elastic scattering of some ions with heavy target in low range of energies.<sup>9</sup>

So far, numerous attempts have been developed to calculate the bound-state energies of the WS potential in the framework of the Schrödinger equation by means of various methods; the Nikiforov–Uvarov method<sup>10–12</sup> and the shape invariance and Hamiltonian hierarchy.<sup>13</sup> However, the solutions in Refs. 10 and 11 are at most valid for  $R_0 = 0$ , in which the WS potential turns out to become the Rosen–Morse-type potential. The two-component approach to the one-dimensional Dirac equation with WS potential is applied to obtain the scattering and bound-state solutions.<sup>14</sup> The exact solution to the relativistic Dirac equation was obtained for a single particle with spin and pseudospin symmetry moving in a central WS potential using two-component spinors.<sup>15–17</sup> An approach has been developed to obtain the bound state solutions of the scattering of a Klein–Gordon particle by a WS potential.<sup>18</sup> Recently, the approximated eigenenergies and eigenfunctions of the Dirac equation for the WS potential and a tensor potential with the arbitrary spin–orbit coupling quantum number  $\kappa$  under pseudospin and spin symmetry have been obtained.<sup>19</sup>

Recently, Ma and Xu have proposed an improved quantization rule (IQR) and shown its power in calculating the energy levels of all bound states for some solvable quantum systems.<sup>20,21</sup> The method has been shown to be effective for calculating the bound state solutions to the Schrödinger and Dirac wave equations with a spherically symmetric potential.<sup>1,22–26</sup> So far, it has been applied, with great success, to study a great number of potentials like the rotating Morse,<sup>22,23</sup> the Kratzer-type,<sup>24</sup> the trigonometric Rosen–Morse,<sup>25</sup> the hyperbolic and the second Pöschl–Teller-like potentials<sup>26</sup> and the Hulthén potential<sup>1</sup> and so forth. Very recently, Gu and Sun<sup>1</sup> have extended the application of the IQR to the solution of the  $D$ -dimensional Schrödinger equation with the Hulthén potential for  $l \neq 0$  using the usual approximation to deal with the centrifugal term.<sup>27–29</sup>

In this paper, we present a new systematical approach to solve the Schrödinger equation in  $D$ -dimensions with WS potential for its energy eigenvalues and the corresponding eigenfunctions by means of the IQR method using the Pekeris approximation scheme<sup>30–32</sup> to deal with the centrifugal term. This alternative approach has recently shown its accuracy in calculating the analytic spectrum of the Hulthén potential for  $l \neq 0$ .<sup>1</sup> Further, we give analytic tests using energy calculations for interdimensional degeneracy, i.e.  $(n, l, D) \rightarrow (n, l \pm 1, D \mp 2)$  corresponding to the confined ( $D = 2 - 4$ )-dimensional WS potential.

This paper is organized as follows. In Sec. 2, the IQR method is reviewed and extended to any arbitrary dimension ( $D \geq 2$ ). In Sec. 3, the  $D$ -dimensional ( $D \geq 2$ ) Schrödinger equation is solved by this method with arbitrary  $l$ -states to obtain all the energy states of the WS potential using the Pekeris approximation. In Sec. 4, we calculate the corresponding hyperradial wave functions of the WS potential. In Sec. 5, the interdimensional degeneracy is introduced. Finally, some concluding remarks are given in Sec. 6.

## 2. Improved Quantization Rule

A brief outline to the improved quantization rule is presented with an extension to the  $D$ -dimensional space ( $D \geq 2$ ). The details can be found in Refs. 20 and 21. The IQR has recently been proposed to solve exactly the one-dimensional (1D) Schrödinger equation:

$$\psi''(x) + k(x)^2\psi(x) = 0, \quad k(x) = \frac{\sqrt{2\mu[E - V(x)]}}{\hbar}, \quad (1)$$

where the prime denotes the derivative with respect to the variable  $x$ . Here  $\mu$  is the reduced mass of the two interacting particles,  $k(x)$  is the momentum and  $V(x)$  is a piecewise continuous real potential function of  $x$ . The Schrödinger equation is equivalent to the Riccati equation

$$\phi'(x) + \phi(x)^2 + k(x)^2 = 0, \quad (2)$$

where  $\phi(x) = \psi'(x)/\psi(x)$  is the logarithmic derivative of wave function  $\psi(x)$ . Due to the Sturm–Liouville theorem, the  $\phi(x)$  decreases monotonically with respect to  $x$  between two turning points, where  $E \geq V(x)$ . Specifically, as  $x$  increases across a node of the wave function  $\psi(x)$ ,  $\phi(x)$  decreases to  $-\infty$ , jumps to  $+\infty$ , and then decreases again.

Moreover, Ma and Xu<sup>20,21</sup> have generalized this exact quantization rule to the three-dimensional (3D) radial Schrödinger equation with spherically symmetric potential by simply making the replacements  $x \rightarrow r$  and  $V(x) \rightarrow V_{\text{eff}}(r)$ :

$$\int_{r_A}^{r_B} k(r)dr = N\pi + \int_{r_A}^{r_B} k'(r) \frac{\phi(r)}{\phi'(r)} dr, \quad k(r) = \frac{\sqrt{2\mu[E_{n,l} - V_{\text{eff}}(r)]}}{\hbar}, \quad (3)$$

where  $r_A$  and  $r_B$  are two turning points determined from the relation  $E_{n,l} = V_{\text{eff}}(r)$ ,  $N = n + 1$  is the number of nodes of  $\phi(r)$  in the region  $E_{n,l} \geq V_{\text{eff}}(r)$  and it is larger by one than the number of nodes of wave function  $\psi(r)$ . The first term  $N\pi$  is the contribution from the nodes of the logarithmic derivative of wave function, and the second term in (3) is called the quantum correction. It is found that, for all well-known exactly solvable quantum systems, this quantum correction is independent of the number of nodes of wave function of the system. This means that it is enough to consider the ground state in calculating the quantum correction, i.e.

$$Q_c = \int_{r_A}^{r_B} k_0'(r) \frac{\phi_0(r)}{\phi_0'(r)} dr = \pi\nu. \quad (4)$$

The quantization rule still holds for Schrödinger equation with spherically symmetric potential in  $D$  dimensions. In what follows, we shall employ this method to solve the Schrödinger equation in  $D$  dimensions with WS potential using the Pekeris approximation to deal with the centrifugal term.

### 3. Eigenvalues of the Woods–Saxon Potential

The  $D$ -dimensional Schrödinger equation with spherically symmetric potential  $V(r)$  for arbitrary  $l$ -state takes the form

$$\left(-\frac{\hbar^2}{2\mu}\nabla_D^2 + V(r) - E_{n,l}\right)\psi_{n,l,m}(r, \Omega_D) = 0, \quad (5)$$

where the representation of the Laplacian operator  $\nabla_D^2$ , in spherical coordinates, is

$$\nabla_D^2 = \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r} \frac{\partial}{\partial r} - \frac{l(l+D-2)}{r^2}, \quad (6)$$

and

$$\psi_{n,l,m}(r, \Omega_D) = \psi_{n,l}(r)Y_l^m(\Omega_D), \quad \psi_{n,l}(r) = r^{-(D-1)/2}u(r), \quad (7)$$

where  $Y_l^m(\Omega_D)$  is the hyperspherical harmonics. The wave functions  $\psi_{n,l,m}(r, \Omega_D)$  belong to the energy eigenvalues  $E_{n,l}$  and  $V(r)$  is taken, in the present study, as the WS potential in the configuration space and  $r$  represents the  $D$ -dimensional intermolecular distance  $(\sum_{i=1}^D x_i^2)^{1/2}$ .

Further, substituting Eqs. (6) and (7) in Eq. (5) yield the wave equation satisfying the radial wave function  $u(r)$  in a simple analogy to the  $2D$  and  $3D$  radial Schrödinger equation

$$\frac{d^2u(r)}{dr^2} + \frac{2\mu}{\hbar^2}[E_{n,l} - V_{\text{eff}}(r)]u(r) = 0, \quad (8)$$

where  $E_{n,l}$  is the bound state energy of the system and  $V_{\text{eff}}(r)$  is the deformed WS<sup>2</sup> effective potential in  $D$  dimensions defined by

$$V_{\text{eff}}(r) = -\frac{V_0 e^{-\frac{(r-R_0)}{a}}}{1 + q e^{-\frac{(r-R_0)}{a}}} + \frac{\lambda}{r^2}, \quad (9)$$

with the parameter

$$\lambda = \frac{(\Lambda^2 - 1)\hbar^2}{8\mu}, \quad \Lambda = 2l + D - 2, \quad (10)$$

where  $R_0 = r_0 A^{1/3}$  is the nuclear radius with  $r_0 = 1.25$  fm (may vary by as much as 0.2 fm depending on the specific nuclide) and  $A$  the atomic mass number of protons ( $Z$ ) plus number of neutrons ( $N$ ) in target nucleus,  $V_0 \approx 50$  MeV (having dimension of energy) is the potential well depth and  $a \approx 0.5$ – $0.6$  fm is the length representing the surface thickness that usually adjusted to the experimental values of ionization energies.<sup>12</sup> In the case of negative eigenenergies (i.e. when  $E_{n,l} \in [-50 \text{ MeV}, 0]$ ), we have the well-known bound-states problem while in the case of positive eigenenergies (i.e. when  $E_{n,l} \in [0, 1000 \text{ MeV}]$ ), we have the well-known resonance problem. It should be noted that the deformation parameter  $q$  is a real parameter and can be taken equal to 1 and  $-1$  for the WS and the Hulthén potentials, respectively in the calculations. The radial wave function  $u(r)$  satisfying Eq. (8) should be normalizable and finite near  $r = 0$  and  $r \rightarrow \infty$  for the bound-state solutions. The

wave equation (8) with the WS potential is an exactly solvable problem for  $l = 0$  (*s*-wave),<sup>3,4,15</sup> however, it cannot be solved analytically for  $\lambda \neq 0$  because of the orbital coupling term  $\lambda r^{-2}$ . Therefore, to solve Eq. (8) analytically, we must use the Pekeris approximation<sup>30–32</sup> to deal with the centrifugal term. By inserting the conversions  $x = (r - R_0)/R_0$  and  $\alpha = R_0/a$ , where  $r \in (0, \infty) \rightarrow x \in (-1, \infty)$ , the following exponential form can be used instead of the centrifugal term

$$\frac{1}{r^2} \approx \frac{1}{R_0^2} \left[ d_0 + d_1 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} + d_2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} \right], \tag{11}$$

which can be expanded around the minimum point  $r \approx R_0$  (or  $x = 0$ ) only up to the second order as

$$\frac{1}{r^2} \approx \frac{1}{R_0^2} \left[ d_0 + \frac{d_1}{2} + \frac{d_2}{4} - \frac{\alpha}{4}(d_1 + d_2)x + \frac{\alpha^2}{16}d_2x^2 - \dots \right], \tag{12}$$

where  $d_0$ ,  $d_1$  and  $d_2$  are coupling constant parameters. It is worth to note that the above expansion is valid for low rotational energy states and  $q = 1$ . The expression  $r^{-2}$  can also be expanded around  $x = 0$  up to the second order term as

$$\frac{1}{r^2} = \frac{1}{R_0^2}(1 - 2x + 3x^2 - \dots). \tag{13}$$

Comparing Eqs. (12) and (13), we obtain the constant parameters for the WS potential as

$$d_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad d_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad d_2 = \frac{48}{\alpha^2}. \tag{14}$$

However, these constant parameters for the Hulthén potential take values as

$$d_0 = \frac{1}{12}, \quad d_1 = 1, \quad d_2 = 1. \tag{15}$$

Further, by defining

$$b = \left( \frac{Q\delta}{R_0} \right)^2, \quad \delta^2 = \frac{\hbar^2}{2\mu} \left( l + \frac{D-1}{2} \right) \left( l + \frac{D-3}{2} \right), \tag{16}$$

where the scaling parameter  $Q$  is introduced with the purpose of obtaining solution for the Hulthén potential as well. So we take  $Q = 1$  for the WS potential but  $Q = \alpha$  and  $R_0 = 1$  for the Hulthén potential.

Equation (8) can be rewritten as

$$\begin{aligned} \frac{d^2 u(x)}{dx^2} + \frac{2\mu R_0^2}{\hbar^2} \left[ E_{n,l} + V_0 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} \right. \\ \left. - b \left( d_0 + d_1 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} + d_2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} \right) \right] u(x) = 0, \end{aligned} \tag{17}$$

where  $n$  and  $l$  signify the radial and orbital angular quantum numbers, respectively.

We now study this quantum system through the improved exact quantization rule. At first, we introduce a new variable

$$y = \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}}, \quad \frac{dy}{dx} = -\alpha y(1 - qy), \quad \frac{dy}{dr} = -\frac{\alpha}{R_0} y(1 - qy), \quad (18)$$

where  $y \in \left(0, \frac{e^\alpha}{1+qe^\alpha}\right)$ . Overmore, the turning points  $y_A$  and  $y_B$  are determined by solving  $V_{\text{eff}}(y) = bd_2y^2 + (bd_1 - V_0)y + bd_0 = E_{n,l}$  as

$$y_A = \frac{V_0}{2bd_2} - \frac{d_1}{2d_2} - \frac{1}{2bd_2} \sqrt{(V_0 - bd_1)^2 + 4bd_2(E_{nl} - bd_0)}, \quad (19a)$$

$$y_B = \frac{V_0}{2bd_2} - \frac{d_1}{2d_2} + \frac{1}{2bd_2} \sqrt{(V_0 - bd_1)^2 + 4bd_2(E_{nl} - bd_0)}, \quad (19b)$$

with the properties

$$y_A + y_B = \frac{V_0}{bd_2} - \frac{d_1}{d_2}, \quad y_A y_B = -\frac{1}{bd_2}(E_{nl} - bd_0). \quad (20)$$

The momentum  $k(y)$  between two turning points is expressed as

$$\begin{aligned} k(y) &= \frac{\sqrt{2\mu}}{\hbar} \sqrt{bd_2} \sqrt{-y^2 + \left(\frac{V_0}{bd_2} - \frac{d_1}{d_2}\right)y + \frac{1}{bd_2}(bd_0 - E_{nl})} \\ &= \frac{\sqrt{2\mu}}{\hbar} \sqrt{bd_2} \sqrt{(y_B - y)(y - y_A)}, \end{aligned} \quad (21)$$

$$\frac{dk(y)}{dy} = \frac{\sqrt{2\mu}}{2\hbar} \sqrt{bd_2} \left( \sqrt{\frac{y_B - y}{y - y_A}} - \sqrt{\frac{y - y_A}{y_B - y}} \right). \quad (22)$$

The Riccati equation (2) now becomes

$$-\frac{\alpha}{R_0} y(1 - qy) \frac{d\phi_0(y)}{dy} = -\frac{2\mu}{\hbar^2} [E_0 - bd_2y^2 + (V_0 - bd_1)y - bd_0] - \phi_0(y)^2, \quad (23)$$

having the only possible first order polynomial solution satisfying

$$\phi_0(r) = c_1 y + c_2, \quad \frac{d\phi_0(r)}{dr} = -\frac{\alpha}{R_0} c_1 y(1 - qy), \quad c_1 > 0, \quad (24)$$

where we have used  $\phi_0(r) \equiv \phi_0(y)$ . Substituting  $\phi_0(y)$  into Eq. (23), one has the ground state wave function and energy eigenvalue solutions

$$\begin{cases} \phi_0(y) = \frac{\alpha}{R_0} my + \left( \frac{\alpha}{2R_0} - \frac{2\mu}{\hbar^2} \frac{(V_0 - bd_1)}{2m\alpha} R_0 \right), \\ m = -\frac{q}{2} \left( 1 \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2 d_2}{\alpha^2 q^2}} \right), \quad m \geq 0, \\ \tilde{E}_{n=0} = E_0 - bd_0 = -\frac{\hbar^2}{2\mu} \left[ \frac{\alpha}{2R_0} - \frac{2\mu}{\hbar^2} \frac{(V_0 - bd_1)}{2m\alpha} R_0 \right]^2. \end{cases} \quad (25)$$

The new quantum number is chosen as  $m = -\frac{q}{2}\left(1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2 d_2}{\alpha^2 q^2}}\right)$  for the WS potential and  $m = -\frac{q}{2}\left(1 + \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2 d_2}{\alpha^2 q^2}}\right)$  for the Hulthén potential. After a lengthy algebra but straightforward, we can calculate the integral of the quantum correction (4) based on the ground state as

$$Q_c = -\pi \left( \frac{\sqrt{2\mu d_2}}{\hbar} \frac{\delta Q}{q \alpha} + \frac{m}{q} + 1 \right). \tag{26}$$

The integral of the momentum  $k(r)$  in the quantization rule (3) is calculated as

$$\begin{aligned} \int_{r_A}^{r_B} k(r) dr &= -\frac{\sqrt{2\mu}}{\hbar} \sqrt{\frac{bd_2}{q^2}} \frac{R_0}{\alpha} \int_{z_A}^{z_B} \left( \frac{\sqrt{(y - y_A)(y_B - y)}}{y} + q \frac{\sqrt{(y - y_A)(y_B - y)}}{1 - qy} \right) dy \\ &= \pi \frac{\sqrt{2\mu}}{\hbar} \frac{R_0}{\alpha} \left( \sqrt{\frac{bd_2}{q^2}} + \sqrt{bd_0 - E_{n,l}} - \sqrt{b \left( d_0 + \frac{d_1}{q} + \frac{d_2}{q^2} \right) - \frac{V_0}{q} - E_{n,l}} \right). \end{aligned} \tag{27}$$

Using the relations (26) and (27), the improved quantization rule (3) turn out to be

$$\begin{aligned} \pi \frac{\sqrt{2\mu}}{\hbar} \frac{R_0}{\alpha} \left( \sqrt{\frac{bd_2}{q^2}} + \sqrt{bd_0 - E_{n,l}} - \sqrt{b \left( d_0 + \frac{d_1}{q} + \frac{d_2}{q^2} \right) - \frac{V_0}{q} - E_{n,l}} \right) \\ = \pi \left( n - \frac{\sqrt{2\mu d_2}}{\hbar} \frac{\delta Q}{q \alpha} - \frac{m}{q} \right). \end{aligned} \tag{28}$$

Thus, based on the improved quantization rule we can finally obtain the approximated energy levels  $E_{n,l}$  valid for both the WS and the Hulthén potentials in the  $D$  dimensions,

$$\begin{aligned} E_{n,l}^{(D)} &= \frac{\delta^2 Q^2}{R_0^2} d_0 - \frac{\hbar^2 R_0^2}{2\mu \alpha^2} \\ &\times \left[ \frac{\alpha^2 \left( 2n + 1 \mp \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2 d_2}{\alpha^2 q^2}} \right)}{4R_0^2} - \frac{\frac{2\mu}{\hbar^2} (qd_1 + d_2) \frac{\delta^2 Q^2}{q^2} - \frac{2\mu V_0}{\hbar^2} \frac{R_0^2}{q}}{R_0^2 \left( 2n + 1 \mp \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2 d_2}{\alpha^2 q^2}} \right)} \right]^2, \end{aligned} \tag{29}$$

where  $n, l = 0, 1, 2, \dots$ . Further, by setting  $Q = 1$  and the deformation parameter  $q = 1$  in the above equation, we get the energy spectrum of the WS potential as

$$\begin{aligned} E_{n,l}^{(D)} &= \frac{\delta^2}{R_0^2} d_0 - \frac{\hbar^2 a^2}{2\mu} \\ &\times \left[ \frac{\left( 2n + 1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{\delta^2 d_2}{\alpha^2}} \right)}{4a^2} - \frac{\frac{2\mu}{\hbar^2} \frac{\delta^2}{R_0^2} (d_1 + d_2) - \frac{2\mu V_0}{\hbar^2}}{\left( 2n + 1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{\delta^2 d_2}{\alpha^2}} \right)} \right]^2, \end{aligned} \tag{30}$$

In three dimensions ( $D = 3$ ), it can be reduced to the form

$$E_{n,l}^{(D)} = \frac{\hbar^2 l(l+1)}{2\mu R_0^2} d_0 - \frac{\hbar^2 a^2}{2\mu} \times \left[ \frac{\left(2n+1 - \sqrt{1+4l(l+1)\frac{d_2}{\alpha^2}}\right)}{4a^2} - \frac{\frac{1}{R_0^2} l(l+1)(d_1+d_2) - \frac{2\mu}{\hbar^2} V_0}{\left(2n+1 - \sqrt{1+4l(l+1)\frac{d_2}{\alpha^2}}\right)} \right]^2, \tag{31}$$

which is essentially same as that obtained by other Nikiforov–Uvarov method.<sup>33</sup>

On the other hand, the case when we set the parameters values  $d_0 = 1/12$ ,  $d_2 = d_3 = 1$ ,  $q = -1$ ,  $Q = \alpha$ ,  $R_0 = 1$  and  $V_0 = \alpha Z e^2$  in Eq. (29), then we obtain the energy spectrum of the Hulthén potential:

$$E_{n,l}^{(D)} = \frac{\hbar^2 \alpha^2}{2\mu} \left\{ \frac{1}{12} \left( l + \frac{D-1}{2} \right) \left( l + \frac{D-3}{2} \right) - \left[ \frac{\mu Z e^2}{\hbar^2 \left( n + l + \frac{D-1}{2} \right) \alpha} - \frac{\left( n + l + \frac{D-1}{2} \right)}{2} \right]^2 \right\}, \tag{32}$$

which is similar to the one obtained in Ref. 1 (if  $d_0 = 0$  in the usual approximation).

#### 4. Eigenfunctions

We are now in the position to study the corresponding eigenfunction of this quantum system for completeness. The Riccati equation of the relation (8) is

$$\phi'(r) = -\frac{2\mu}{\hbar^2} [E_{n,l} - V_{\text{eff}}(r)] - \phi(r)^2, \tag{33}$$

where

$$\phi(r) = \frac{u'(r)}{u(r)}. \tag{34}$$

Based on

$$u(r) = e^{\int^r \phi(z) dz} = e^{-a \int^r \frac{1}{y(1-xy)} \phi(z) dz}, \tag{35}$$

and using Eq. (24), we can easily calculate the eigenfunction of the ground state as

$$u_0(r) = N_0 \left( e^{-\frac{(r-R_0)}{a}} \right)^{\tilde{\epsilon}_0} \left( 1 + q e^{-\frac{(r-R_0)}{a}} \right)^\nu, \quad \tilde{\epsilon}_0 > 0, \quad \nu \geq 1, \tag{36}$$

where

$$\tilde{\epsilon}_{n=0} = a \sqrt{\frac{2\mu}{\hbar^2} (bd_0 - E_0)}, \quad \nu = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{Q^2 \delta^2}{q^2 \alpha^2} d_2} \right), \tag{37}$$

and  $N_0$  is the normalization constant.



Let us find the eigenfunction for any quantum state  $n$  other than ground state. By considering the boundary conditions

$$y = \begin{cases} 0 & \text{when } r \rightarrow \infty, \\ 1 & \text{when } r \rightarrow 0, \end{cases} \tag{38}$$

with  $u(y) \rightarrow 0$ , we may define a more general radial eigenfunctions, valid for any quantum number  $n$ , of the form:

$$u(y) = y^{\tilde{\epsilon}_{n,l}}(1 + qy)^\nu f(y), \quad y = e^{-\frac{(r-R_0)}{a}}, \quad \tilde{\epsilon}_{n,l} > 0, \quad \nu \geq 1. \tag{39}$$

Substituting Eq. (36) in Eq. (8) leads to the hypergeometric equation

$$y(1 + qy)f''(y) + [1 + 2\tilde{\epsilon}_{n,l} + q(1 + 2\tilde{\epsilon}_{n,l} + 2\nu)y]f'(y) - \left[ \frac{2\mu}{\hbar^2}a^2(bd_1 - V_0) - q\nu(1 + 2\tilde{\epsilon}_{n,l}) \right]f(y) = 0, \tag{40}$$

with the conditions

$$\tilde{\epsilon}_{n,l}^2 + \frac{2\mu}{\hbar^2}a^2(E_{n,l} - bd_0) = 0, \tag{41a}$$

$$q^2\nu(\nu - 1) - \frac{2\mu}{\hbar^2}a^2bd_2 = 0, \tag{41b}$$

which leads to the results

$$\tilde{\epsilon}_{n,l} = a\sqrt{\left(l + \frac{D-1}{2}\right)\left(l + \frac{D-3}{2}\right)\frac{Q^2d_0}{R_0^2} - \frac{2\mu}{\hbar^2}a^2E_{n,l}}, \tag{42a}$$

$$\nu = \frac{1}{2}(1 \pm \zeta), \quad \zeta = \sqrt{1 + 4\left(l + \frac{D-1}{2}\right)\left(l + \frac{D-3}{2}\right)\frac{Q^2}{q^2\alpha^2}d_2}, \tag{42b}$$

where  $+$  and  $-$  are taken for the Hulthén and the WS potentials, respectively. The solution to Eq. (40) is then

$$F(y) = {}_2F_1(A, B; C; y) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)} \sum_{k=0}^{\infty} \frac{\Gamma(A+k)\Gamma(B+k)}{\Gamma(C+k)} \frac{y^k}{k!}, \tag{43}$$

where

$$\begin{aligned} A &= \tilde{\epsilon}_{n,l} - q\nu - \sqrt{U} = -n, \quad n = 0, 1, 2, \dots, \\ B &= \tilde{\epsilon}_{n,l} - q\nu + \sqrt{U}, \quad C = 1 + 2\tilde{\epsilon}_{n,l}, \end{aligned} \tag{44}$$

where  ${}_2F_1(A, B; C; y)$  is the hypergeometric function. Now, we may write down the radial wave functions (34) as

$$\begin{aligned} u_{n,l}(r) &= \mathcal{N}_{nl} \left( e^{-\frac{(r-R_0)}{a}} \right)^{\tilde{\epsilon}_{n,l}} \left( 1 + qe^{-\frac{(r-R_0)}{a}} \right)^\nu \\ &\quad \times {}_2F_1 \left( -n, n + 2\tilde{\epsilon}_{n,l} + q\zeta - q; 1 + 2\tilde{\epsilon}_{n,l}; e^{-\frac{(r-R_0)}{a}} \right). \end{aligned} \tag{45}$$

If now, we set  $n = 0$  in Eq. (45), then we can easily obtain Eq. (36). Finally, the unnormalized total wave functions are obtained as

$$\psi_{n,l,m}(r, \Omega_D) = \mathcal{N}_{nl} r^{-(D-1)/2} \left( e^{-\frac{(r-R_0)}{a}} \right)^{\tilde{\epsilon}_{n,l}} \left( 1 + q e^{-\frac{(r-R_0)}{a}} \right)^\nu \times {}_2F_1 \left( -n, n + 2\tilde{\epsilon}_{n,l} + q\zeta - q; 1 + 2\tilde{\epsilon}_{n,l}; e^{-\frac{(r-R_0)}{a}} \right) Y_l^m(\Omega_D), \quad (46)$$

which is identical to Eq. (42) of Ref. 33 when  $D = 3$ . Thus, the Jacobi polynomials can be expressed in terms of the hypergeometric functions<sup>34</sup>

$$P_n^{(A,B)}(1 - 2x) = \frac{\Gamma(n + 1 + A)}{n! \Gamma(1 + A)} {}_2F_1(-n, n + A + B + 1; A + 1; x). \quad (47)$$

It is worth noting that the hypergeometric function  ${}_2F_1(A, B; C; x)$  is a special case of the generalized hypergeometric function<sup>34</sup>

$${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_1, \dots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k x^k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k k!}, \quad (48)$$

where the Pochhammer symbol is defined by  $(y)_k = \Gamma(y + k)/\Gamma(y)$ .

### 5. Interdimensional Degeneracy

From Eq. (30), it can be seen that two interdimensional states are degenerate whenever<sup>35</sup>

$$(n, l, D) \rightarrow (n, l \pm 1, D \mp 2) \Rightarrow E_{n,l}^{(D)} \rightarrow E_{n,l \pm 1}^{(D \mp 2)}, \quad (49)$$

i.e.

$$E_{n,l}^{(D)} = E_{n,l \pm 1}^{(D \mp 2)} = \frac{\delta^2}{R_0^2} d_0 - \frac{\hbar^2 a^2}{2\mu} \times \left[ \frac{\left( 2n + 1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{\delta^2}{\alpha^2} d_2} \right)}{4a^2} - \frac{\frac{2\mu}{\hbar^2} \frac{\delta^2}{R_0^2} (d_1 + d_2) - \frac{2\mu}{\hbar^2} V_0}{\left( 2n + 1 - \sqrt{1 + \frac{8\mu}{\hbar^2} \frac{\delta^2}{\alpha^2} d_2} \right)} \right]^2. \quad (50)$$

Thus, a knowledge of  $E_{n,l}^{(D)}$  for  $D = 2$  and  $D = 3$  provides the information necessary to find  $E_{n,l}^{(D)}$  for other higher dimensions.

For example,  $E_{0,4}^{(2)} = E_{0,3}^{(4)} = E_{0,2}^{(6)} = E_{0,1}^{(8)}$ . This is the same transformational invariance described for bound states of free atoms and molecules<sup>36–38</sup> and demonstrates the existence of interdimensional degeneracies among states of the confined Hulthén potential.

### 6. Conclusions

By using the improved quantization rule, we have given an alternative method to obtain the approximated energy eigenvalues and eigenfunctions of the Schrödinger equation in  $D$  dimensions with the WS potential for all  $l$ -states within the Pekeris approximation to deal with centrifugal term. We emphasize that the expressions obtained for the energy eigenvalues and eigenfunctions are valid for all real values of  $R_0$ . However, the solutions provided in Refs. 10 and 11 are at most valid for  $R_0 = 0$  (Rosen–Morse-type potential and not the WS potential). The advantage of this method is that it gives the eigenvalues through the calculation of two integrations (26) and (27) and solving the resulting algebraic equation. First, we can easily obtain the quantum correction by only considering the solution of the ground state of quantum system since it is independent of the number of nodes of wave function for exactly solvable quantum system. Second, the wave functions have also been obtained by solving the Riccati equation. The general expressions obtained for the energy eigenvalues and wave functions can be easily reduced to the WS and the Hulthén potentials, the three-dimensional space ( $D = 3$ ),  $s$ -wave ( $l = 0$ ) cases. The method presented here is a systematic one, simple, practical and powerful than the other known methods. Finally, the simplicity of the method motivates us to solve the Dirac equation for this WS potential to see the resulting relativistic effects.

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### Appendix A. Integral Formulas

The following integral formulas are useful during the calculation of the momentum integral and the quantum correction term:

$$\int_{r_A}^{r_B} \frac{r}{\sqrt{(r - r_A)(r_B - r)}} dr = \frac{\pi}{2}(r_A + r_B), \tag{A.1}$$

$$\int_{r_A}^{r_B} \frac{1}{r\sqrt{(r - r_A)(r_B - r)}} dr = \frac{\pi}{\sqrt{r_A r_B}}, \tag{A.2}$$

$$\int_{r_A}^{r_B} \frac{1}{\sqrt{(r - r_A)(r_B - r)}} dr = \pi, \tag{A.3}$$

$$\int_{r_A}^{r_B} \frac{1}{r} \sqrt{(r - r_A)(r_B - r)} dr = \pi \left[ \frac{1}{2}(r_A + r_B) - \sqrt{r_A r_B} \right], \tag{A.4}$$

$$\int_{r_A}^{r_B} \frac{1}{(a + br)\sqrt{(r - r_A)(r_B - r)}} dr = \frac{\pi}{\sqrt{(a + br_A)(a + br_B)}}, \quad r_B > r_A > 0. \tag{A.5}$$

## References

1. X.-Y. Gu and J.-Q. Sun, *J. Math. Phys.* **51**, 022106 (2010).
2. R. D. Woods and D. S. Saxon, *Phys. Rev.* **95**, 577 (1954).
3. A. K. Bose, *Nuovo Cimento* **32**, 679 (1964).
4. S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1974).
5. J. M. G. Gómez, K. Kar, V. K. B. Kota, R. A. Molina and J. Retamosa, *Phys. Lett. B* **567**, 251 (2003).
6. S. E. Massen and C. P. Panos, *Phys. Lett. A* **246**, 530 (1997).
7. B. A. Kotsos and M. E. Grypeos, *Physica B* **229**, 173 (1997).
8. H. Nicolai, *J. Phys. A* **9**, 1497 (1976).
9. O. V. Bessulova, E. A. Romanovsky and T. I. Spasskaya, *J. Phys. G* **29**, 1193 (2003).
10. C. Berkdemir, A. Berkdemir and R. Sever, *Phys. Rev. C* **72**, 027001 (2005).
11. C. Berkdemir, A. Berkdemir and R. Sever, *Phys. Rev. C* **74**, 039902(E) (2006).
12. S. M. Ikhdair and R. Sever, *Int. J. Theor. Phys.* **46**, 1643 (2007).
13. C. Berkdemir, A. Berkdemir and R. Sever, *J. Math. Chem.* **43**, 944 (2008).
14. P. Kennedy, *J. Phys. A* **35**, 689 (2002).
15. C. Berkdemir, A. Berkdemir and R. Sever, *J. Phys. A* **39**, 13455 (2006).
16. J.-L. Tian, N. Wang and Z.-X. Li, *Chinese Phys. Lett. A* **24**, 905 (2007).
17. J.-Y. Guo and Z.-Q. Sheng, *Phys. Lett. A* **338**, 90 (2005).
18. C. Rojas and V. M. Villalba, *Phys. Rev. A* **71**, 052101 (2005).
19. O. Aydođdu and R. Sever, *Eur. Phys. J. A* **43**, 73 (2010).
20. Z. Q. Ma and B. W. Xu, *Europhys. Lett.* **69**, 685 (2005).
21. Z. Q. Ma and B. W. Xu, *Int. J. Mod. Phys. E* **14**, 599 (2005).
22. W.-C. Qiang and S.-H. Dong, *Phys. Lett. A* **363**, 169 (2007).
23. W.-C. Qiang, R.-S. Zhou and Y. Gao, *J. Phys. A: Math. Theor.* **40**, 1677 (2007).
24. S. M. Ikhdair and R. Sever, *J. Math. Chem.* **45**, 1137 (2009).
25. Z.-Q. Ma, A. Gonzalez-Cisneros, B.-W. Xu and S.-H. Dong, *Phys. Lett. A* **371**, 180 (2007).
26. S.-H. Dong and A. Gonzalez-Cisneros, *Ann. Phys.* **323**, 1136 (2008).
27. R. L. Greene and C. Aldrich, *Phys. Rev. A* **14**, 2363 (1976).
28. S. M. Ikhdair and R. Sever, *J. Math. Chem.* **42**, 461 (2007).
29. N. Saad, *Phys. Scr.* **76**, 623 (2007).
30. C. L. Pekeris, *Phys. Rev.* **45**, 98 (1934).
31. C. Berkdemir and J. Han, *Chem. Phys. Lett.* **409**, 203 (2005).
32. C. Berkdemir, *Nucl. Phys. A* **770**, 32 (2006).
33. S. M. Ikhdair and R. Sever, *Cent. Eur. J. Phys.* **8**, 652 (2010).
34. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th edn. (Academic, New York, 1994).
35. H. E. Montgomery, Jr., N. A. Aquino and K. D. Sen, *Int. J. Quantum Chem.* **107**, 798 (2007).
36. D. R. Herrick, *J. Math. Phys.* **16**, 281 (1975).
37. D. R. Herrick and F. H. Stillinger, *Phys. Rev. A* **11**, 42 (1975).
38. D. D. Fratz and D. R. Herschbach, *J. Chem. Phys.* **92**, 6668 (1990).