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Analysis of the Heat Equation with Non-Local Radiation Terms in a Non-Convex Diffuse and Grey Surfaces

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Abstract

In this article we consider heat transfer in a non-convex system that consists of a union of finitely many opaque, conductive and bounded objects which have diffuse and grey surfaces and are surrounded by a perfectly transparent and non-conducting medium (such as vacuum). The resulting problem is non-linear and in general is non-coercive due to the non-locality of the boundary conditions. We discuss the solvability of the problem by proving the existence of a weak solution. We extend the analysis to address the parabolic case and to the case with non-linear material properties. Also we consider some cases when coercivity is obtained and state the corresponding stronger existence results.

Keywords: heat radiation, Stefan-Boltzmann law, non-local boundary value problem.

1. Introduction

Radiative heat exchange is a very important phenomenon in our modern technology. It has to be taken into account in general always when the temperature on a visible surface of the system is high enough or when other heat transfer mechanisms are not present (like in vacuum, for example). A part from some simple cases like convex radiating enclosure and known irradiation from infinity, we have to take into account the radiative heat exchange between different parts of the surface of our system. This leads to non-local boundary conditions on radiating part of the boundary. In some industrial applications involving heat radiation the material surfaces are not perfectly black, which implies that part of the heat radiation falling on the surface is reflected. To simplify the treatment of the reflection we will assume that the surfaces are diffuse emitters and reflectors (i.e. they emit and reflect radiation uniformly to all directions). Moreover, we assume that the surfaces are grey, that is, they emit and absorb all wavelengths in the same manner. This means that we can forget the wavelengths spectrum (colour) of the radiation and model only the total intensity of the radiated waves. In this work we will consider modeling of heat transfer in a system that consists of a union of finitely many opaque, conductive and bounded enclosures which have diffuse and grey surfaces and are surrounded by perfectly transparent and non-conducting medium (like vacuum). The organization of the paper is as follows:

In section 2, we give a short description of the mathematical model for the heat transfer in a non-convex enclosure. This leads to a non-local problem.

In section 3, we will confirm that the variational form for our problem is well defined in some appropriate spaces. This can be achieved by studying the properties of the integral operator.

In section 4, we discuss the solvability of the boundary value problem where the main difficulties are the lack of monotonicity and coercivity.

In section 5, we extend the analysis to the parabolic case and to the case with non-linear material properties (temperature dependent conductivity or emissivity). We also mention some cases where the coercivity is obtained and stating the corresponding stronger existence results.

Throughout the work we shall use the standard Lebesgue and Sobolev spaces with the following notations: By $L^p(S)$ we denote the space of measurable functions φ for which $\|\varphi\|_{L^p} = \left(\int_s |\varphi|^p \right)^{\frac{1}{p}}$ is finite. By $W^{m,p}(S)$ we denote the space of $L^p(S)$ functions whose generalized derivatives up to order m are in $L^p(S)$. The space $W^{m,p}(S)$ is equipped with the norm $\|\varphi\|_{m,p} = \sum_{k=0}^{m} \|D^k \varphi\|_{L^p}$. Notation $H^m(S)$ with norm $\|\varphi\|_m$ is used for $W^{m,2}(S)$ and finally $H^1(\Omega;S)$ stands for the space $\{\varphi \in \mathrm{H}^1(\Omega) \,|\, \varphi = 0 \;\; on \;\, S\}.$.

2. Heat transfer model

We consider heat transfer in a system that consists of a union of finitely many opaque, conductive and bounded objects which have diffuse and gray surfaces and are surrounded by a perfectly transparent and non-conducting medium (such as vacuum). We denote the union of conductive objects by Ω and note that in general Ω is not a connected set. We assume that the boundary $\partial\,\Omega$ of Ω can be represented by $\partial\,\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma$ where Γ_0 denotes the part of the boundary in which the temperature is given. On Γ_1 the heat flux is given. Γ_2 denotes the set $\partial\,\Omega \cap \partial\,\Omega_c \setminus (\Gamma_0 \cap \Gamma_1)$ where Ω_c is the convex hull of Ω. The set Γ forms the rest of $\partial\,\Omega$. . By definition any point on Γ is the interior point of Ω_c . Consequently it will 'see' some other points of Γ , that is, for any point $x \in \Gamma$ there exists a set $\Gamma_x \subset \Gamma$ defined by $\Gamma_x = \left\{ x \in \Gamma \,\middle|\, \overline{xy} \cap \Omega = \varphi \right\}$. Obviously, $y \in \Gamma_x$ implies that $x \in \Gamma_y$. Thus Γ can be decomposed into disjoint components Γ_i. Under the above assumptions, the stationary heat equation for Ω combined with the heat radiation on $\Gamma_2 \cup \Gamma$ results in a coupled system for the absolute temperature T and the intensity q of the total radiation that leaves the surface. Thus, we have

$$-k\,\Delta T + v \cdot \nabla T = f \quad in \quad \Omega \tag{2.1}$$

where k is the heat conductivity coefficient, v is the convection velocity multiplied by the heat capacity and f is the internal heat source. On the boundary Γ_0 we have the condition

$$T = T_0 \qquad on \;\; \Gamma_0 \tag{2.2}$$

where T_0 is the effective external radiation temperature.

On Γ_2 the body radiates heat to infinity, which results to heat balance

$$k\frac{\partial T}{\partial n} + \varepsilon\,\sigma\,T^4 = \varepsilon\,q^{\infty} \quad on \;\; \Gamma_2 \tag{2.3}$$

where q^{∞} is the intensity of radiation coming from outside the system. By q^{∞} we denote the emissivity of the surface. Finally, on Γ_i we have

$$k \frac{\partial T}{\partial n} + \left(I - K_i\right) q_i = K_i^{\infty} q^{\infty} \quad on \ \Gamma_i \tag{2.4}$$

Where K_i^{∞} denotes the amount of outside radiation that reaches Γ_i. q_i is the radiosity of the surface Γ_i . It depends on the surface temperature through the relation

$$\varepsilon \sigma T^4 + \left(I - (1-\varepsilon) K_i\right) q_i = (1-\varepsilon) K_i^{\infty} q^{\infty} \ on \ \Gamma_i \tag{2.5}$$

In fact equation (2.5) can be written as

$$q_i = (1-\varepsilon)\left(K_i q_i + K_i^{\infty} q^{\infty}\right) + \varepsilon \sigma T^4 \tag{2.6}$$

Equation (2.6) states that the emitted radiation is the sum of the reflected radiation and the Stefan-Boltzmann radiation on the surface itself. Due to the non-convexity of the surface Γ_i we have to take into account the self illumination which is described by the integral operator K_i. The three dimensional explicit form of K_i is known [1, 8]. It reads

$$K_i \, q(x) = \int_{\Gamma_i} G(x, y) \, q(y) \, dy \tag{2.7}$$

where

$$G(x, y) = G^*(x, y) \, \beta(x, y) \tag{2.8}$$

and

$$G^*(x, y) = \frac{n_x \cdot (y - x) \ n_y \cdot (x - y)}{\pi \ |y - x|^4} \ . \tag{2.9}$$

Here n_x is the inner normal to Γ_i at the point x. n_y is defined analogously. The function $\beta(x, y)$ is called the visibility function. More precisely, if the point x and y 'see each other' along a straight line segment that does not intersect Γ_i at any other point, then $\beta(x, y) = 1$; otherwise $\beta(x, y) = 0$. Before introducing the variational form note that the Stefan-Boltzmann law is physically meaningful only for positive temperatures then we can monotonize it by replacing σT^4 by $h(T) = \sigma |T|^3 T$. To this end, the weak formulation of (2.1) - (2.5) reads as

$$\int_{\Omega} k \nabla T \nabla \varphi + v . \nabla T \varphi + \int_{\Gamma_2} \varepsilon h(T) \varphi + \sum_i \int_{\Gamma} (I - K_i) q_i \varphi = \langle f, \varphi \rangle_{X \times X} \quad \forall \varphi \in X \tag{2.10}$$

$$- \int_{\Gamma_i} \varepsilon h(T) \psi_i + \int_{\Gamma_i} \left(I - (1-\varepsilon) K_i\right) q_i \psi_i = \langle g, \psi_i \rangle_{W_i' \times W_i} \quad \forall \psi_i \in W_i \tag{2.11}$$

for the radiosity q and the temperature T. The system (2.10) - (2.11) is well defined (in three-dimensional case) if we set $X = H^1(\Omega; \Gamma_0) \cap L^5(\Gamma_2) \cap_i L^5(\Gamma_i)$ and $W_i = L^5(\Gamma_i)$ (see the introduction for the definition of the spaces). Then the radiosity q_i will be in $L^{5/4}(\Gamma_i)$ and T in X .

3. Radiation on non-convex surfaces

In this section we recall some properties of the operator K_i defined in (2.7) and the corresponding kernel $G(x, y)$ defined in (2.8) - (2.9).

Lemma1. Let Γ_i be a Ljapunow surface in $C^{1,\delta}$ with $\delta \in [0,1)$. Then for any arbitrary point $x \in \Gamma_i$ we obtain

$$\int_{\Gamma_i} G(x, y)\, d\Gamma_y = 1.$$

Proof: See [5 , 6].

Lemma 2. The operator K_i is non-negative, i.e. for the integral kernel $G(x, y)$ it holds $G(x, y) \geq 0$. The mapping $K_i : L^p(\Gamma_i) \rightarrow L^p(\Gamma_i)$ is compact for $1 \leq p \leq \infty$. Furthermore, we get

(a) $\|K_i\| = 1$ in L^p for $1 \leq p \leq \infty$,

(b) The spectral radius $\rho\,(K_i) = 1$.

Proof: See [5, 6].

Let us now consider equation (2.5) or its weak formulation (2.10). We denote by E the operator corresponding to multiplication with ε then it holds that q_i can be solved from (2.5), thus we have

Lemma 3. The operator $\mathrm{I} - (\mathrm{I} - \mathrm{E})\, K_i$ from $L^p(\Gamma_i)$ into itself is invertible and its inverse is nonnegative.

Proof: See [4] .

Due to Lemma 3 the problem (2.10) - (2.11) can be written in terms of T alone by solving q_i from (2.11). If we introduce the operators

$$A_i = (\mathrm{I} - K_i)(\mathrm{I} - (\mathrm{I} - \mathrm{E})\, K_i)^{-1}\mathrm{E} \tag{3.1}$$

$$= (\mathrm{I} - \mathrm{E}\, K_i)(\mathrm{I} - (\mathrm{I} - \mathrm{E})\, K_i)^{-1}\mathrm{E} \tag{3.2}$$

$$= \mathrm{E}\left(\mathrm{I} - \left(\mathrm{I} - K_i (\mathrm{I} - \mathrm{E})^{-1} K_i\, \mathrm{E}\right)\right) \tag{3.3}$$

Then $(\mathrm{I} - K_i)\, q_i = A_i\, h\,(T) + \widetilde{g}$ and the problem reads as

$$\int_{\Omega} k \nabla T \nabla \varphi + v \cdot \nabla T\, \phi + \int_{\Gamma_2} \varepsilon\, h(T)\, \phi + \sum_i \int_{\Gamma_i} A_i\, h(T)\, \phi = \left\langle \widetilde{f}, \phi \right\rangle_{X' \times X} \tag{3.4}$$

For the operator A_i we have

Lemma 4 . The symmetric part of operator A_i, $A_i + A_i^T$ from $L^p(\Gamma_i)$ into itself is positively semi definite.

Proof: Let $u \in L^p(\Gamma_i)$ be arbitrary. Suppose that $\widetilde{u}$ is the solution of $(\mathrm{I} - (\mathrm{I} - \mathrm{E})\, K)\, \widetilde{u} = \mathrm{E}\, u$, then

$$\left(u, \left(A_i + A_i^T\right) u\right) = 2\left(\mathrm{E}^{-1} (\mathrm{I} - (\mathrm{I} - \mathrm{E})\, K_i)\, \widetilde{u}, (\mathrm{I} - K_i)\, \widetilde{u}\right)$$

$$= 2\, \left(\widetilde{u}, (\mathrm{I} - K_i)\, \mathrm{E}^{-1}\, (\mathrm{I} - K_i)\, \widetilde{u}\right) + 2\left(\widetilde{u}, \left(K_i - K_i^2\right) \widetilde{u}\right) \geq 0$$

as $\rho\,(K_i) = 1$.

Lemma 5. The operator A_i can be written as $A_i = \mathrm{I} - \mathrm{B}_i$ where $\mathrm{B}_i \geq 0$ and $\rho\,(\mathbf{B}_i) \leq 1$. Furthermore if ε is a constant, then $\|\mathrm{B}_i\| \leq 1$ as a mapping from $L^p(\Gamma_i)$ into itself. Strict inequality is obtained when ε is constant and $\|\mathrm{K}_i\| < 1$.

Proof: [4]

4. Existence of solutions

Before discussing some partial coercivity results, we can write problem (3.4) into the form

$$\tilde{a}(\mathrm{T},\varphi)=a_1(\mathrm{T},\varphi)+a_2(\mathrm{T},\varphi)+\sum_i a_i(\mathrm{T},\varphi)=\langle f,\varphi\rangle \quad \forall\varphi\in X \tag{4.1}$$

where

$$a_1(\mathrm{T},\varphi)=\int_\Omega k\nabla\mathrm{T}\nabla\varphi+\nu\cdot\nabla\mathrm{T}\varphi\,, \tag{4.2}$$

$$a_2(\mathrm{T},\varphi)=\int_{\Gamma_2}\varepsilon\, h\,(\mathrm{T})\,\varphi, \tag{4.3}$$

$$a_i(\mathrm{T},\varphi)=\int_{\Gamma_i} A_i\, h(\mathrm{T})\,\varphi\,. \tag{4.4}$$

Lemma 6. Suppose that Ω is bounded and connected, $\nabla\cdot\nu=0$, $\nu\cdot n\geq 0$ on $\partial\Omega\setminus\Gamma_0$ and one of the following conditions holds,

(i) Γ_0 has positive surface measure

(ii) Γ_2 has positive surface measure

(iii) Γ_i has positive surface measure and A_i can be written as $A_i=\mathrm{I}-\mathrm{B}_i$ with $\|\mathrm{B}_i\|\leq 1$. Then the form a_1 is coercive in $H^1(\Omega)$ or a_1+a_2 is coercive in $H^1(\Omega)\mathrm{I}\ L^5(\Gamma_2)$ or a_1+a_i is coercive in $H^1(\Omega)\mathrm{I}\ L^5(\Gamma_i)$.

Proof: Let us first consider the convection term. Integrating by parts we get

$$\int_\Omega \nu\cdot\nabla T\,T\,dx=-\int_\Omega \nu\cdot\nabla T\,T\,dx-\int_\Omega\nabla\cdot\nu\,T^2\,dx+\int_{\partial\Omega}\nu\cdot n\,T^2\,ds=\frac{1}{2}\int_{\partial\Omega}\nu\cdot n\,T^2\,ds\geq 0\,,$$

as T = 0 on Γ_0 and $\nu\cdot n\geq 0$ outside Γ_0. The case (i) follows directly from Poincare inequality [2]. In case (ii) we have

$$a_1(T,T)+a_2(T,T)\geq\int_\Omega k\,\nabla T\,\nabla T+\int_{\Gamma_2}\varepsilon\, h(T)\,T\geq c\left(\|\nabla T\|^2_{L^2(\Omega)}+\|T\|^5_{L^5(\Gamma_2)}\right) \tag{4.5}$$

Since Ω and Γ_2 are bounded then

$$\|\,\mathrm{T}\,\|_{L^2(\Gamma_2)}\leq c\,\|\,\mathrm{T}\,\|_{L^5(\Gamma_2)} \tag{4.6}$$

for some constant c depending on the measure of Γ_2.

This implies that $\|u\|=\|\nabla u\|_{L^2(\Omega)}+\|u\|_{L^5(\Gamma_2)}$ is an equivalent norm in $H^1(\Omega)\mathrm{I}\ L^5(\Gamma_2)$, (see [2]).

On the other hand we have $\dfrac{\tilde{a}(T,T)}{\|T\|}\to\infty$ as $\|\mathrm{T}\|\to\infty$.

Finally for case (iii) we have

$$a_1(T,T)+a_i(T,T)\geq\int_\Omega k\nabla T\,\nabla T+\int_{\Gamma_i}h(T)T-\int_{\Gamma_i}B_i h(T)T\geq c\,\|\,\nabla T\,\|^2_{L^2(\Omega)}+\left(1-\|\,B_i\,\|\right)\sigma\|\,T\,\|^5_{L^5(\Gamma_i)}\,.$$

Here we use the fact that $\|\,h(T)\,\|_{L^{5/4}(\Gamma_i)}=\sigma\|\,T\,\|^4_{L^5(\Gamma_i)}$. Hence the conclusion follows as above.

As we have neither coercivity nor monotonicity results in the general case we will make use of the technique of sub-and supersolutions. For this reason, we denote by X^+ the cone of non-negative elements

$$\mathrm{X}^+=\{\,x\in\mathrm{X}\;:x\geq 0\} \tag{4.7}$$

Theorem 1. Let Ω be a three-dimensional domain with a piecewise $C^{1,\delta}$ -boundary with $\delta \in (0\,,1)$ that satisfies the Lipschitz condition. Assume that $\widetilde{f} \in X'$, the velocity v satisfies the conditions of Lemma 6, and that there exist two functions $\varphi \le \psi$, $\varphi, \psi \in H^1(\Omega) \bigcap_i \overset{5}{L}(\Gamma_i)$, such that

$$\widetilde{a}(\varphi,\omega) \quad \le \quad <\widetilde{f},\omega> \quad \forall\, \omega \in X' \tag{4.8}$$

$$\widetilde{a}(\psi,\omega) \quad \ge \quad <\widetilde{f},\omega> \quad \forall\, \omega \in X' \tag{4.9}$$

Then (4.1) has a solution T. Moreover, $\varphi \le T \le \psi$ in Ω.

Proof: Using Lemma 5 we can rewrite the problem (4.1) into the form

$$a_1(T,\omega) + a_2(T,\omega) + b\,(T,\omega) - c\,(T,\omega) = <\widetilde{f},\omega> \quad \forall\, \omega \in X\,, \tag{4.10}$$

where

$b(T,\omega) = \sum_i \int_{\Gamma_i} h(T)\,\omega$ and $c(T,\,\omega) = \sum_i \int_{\Gamma_i} h(T)\,B^*(\omega)$.

Set now $T_1 = \psi$ and construct a sequence T_n , $n \ge 1$ as follows:

$$a_1(T_n,\omega) + a_2(T_n,\omega) + b(T_n,\omega) = c(T_{n-1},\omega) + <\widetilde{f},\omega> \quad \forall\, \omega \in X\,, n > 1 \tag{4.11}$$

In fact T_n is a decreasing sequence of supersolutions that is bounded from below. To show this, we assume that T_n is a supersolution. Then

$$\begin{aligned} &a_1(T_{n+1},\omega) + a_2(T_{n+1},\omega) - a_1(T_n,\omega) - a_2(T_n,\omega) + b(T_{n+1},\omega) - b(T_n,\omega) \\ &= <\widetilde{f},\omega> - a_1(T_n,\omega) - a_2(T_n,\omega) - b(T_n,\omega) + c(T_n,\omega) \quad \le 0 \quad \forall\, \omega \in X^+ \end{aligned} \tag{4.12}$$

If we choose $\omega = [T_{n+1} - T_n]^+$ then we obtain from above that

$$a_1(\omega,\omega) + \int_{\Gamma_2} 4\,\varepsilon\,\sigma \left|\widetilde{T}\right|^3 \omega^2 + \sum_i \int_{\Gamma_i} 4\sigma \left|\widetilde{T}\right|^3 \omega^2 \quad \le 0\,, \tag{4.13}$$

where $\widetilde{T}$ is between T_n and T_{n+1} . Clearly the left hand side of (4.13) is coercive in $H^1(\Omega)$ and we conclude that $\omega = 0$ and consequently $T_{n+1} \le T_n$. On the other hand c is defined using a non-negative kernel then we see that $c(T_{n+1},\omega) \le c(T_n,\omega) \quad \forall\, \omega \in X^+$.

Hence, it is easy to see that T_{n+1} is also a supersolution . The case of subsolution is treated analogously. Furthermore, if the initial approximation is bounded from below by a subsolution, the later iterates will be also bounded from above. This means that the sequence T_n converges monotonically to a limit which clearly solves the problem (4.1).

5. Extension of analysis

5.1 Coercive cases

Here we consider two important cases when the problem is coercive. The first case is related to two-dimensional problem. Here the trace theorem implies that the boundary values of $H^1(\Omega)$ - functions are in $L^p(\Gamma)$ for all finite p . Thus, the non-linear and non-local boundary terms are well defined in a two-dimensional setting for standard H^1 - functions. Moreover, the mapping from $H^1(\Omega)$ to $L^5(\Gamma)$ is compact. Hence, the boundary terms can be considered in some sense as small perturbations of the elliptic main part. Of course, in 2D the definition of the kernel $\overset{*}{G}(x,y)$ is given [1].

Theorem 2. Suppose that Ω is a connected piecewise $C^{1,\delta}$ - domain in R^2 which satisfies one of the conditions in Lemma 6. Then, if $\tilde{f} \in (\mathrm{H}^1(\Omega))'$ and on all Γ_i we have $a_i(\mathrm{T},\mathrm{T}) \geq 0 \ \forall\, \mathrm{T}$, then (4.1) has a solution.

Proof: (see [4] for details) As $a_i(\mathrm{T},\mathrm{T}) \geq 0 \quad \forall\, i$, the form $\tilde{a}$ is coercive in $\mathrm{H}^1(\Omega)$. The non-local terms are compact. So as the other terms are monotone, we see that $\tilde{a}$ is in fact also pseudo-monotone. Hence the problem (4.1) has at least one solution [9 , 10]. Moreover, the nonnegativity condition for a_i is verified at least when ε is constant on Γ_i , see Lemma 5.

The second case when coerciveness can be achieved is the situation where the radiating surfaces emit part of the radiation out of the system. This makes the non-local operator contractive and results in coerciveness with respect to the $L^5(\Gamma)$ - norm. More precisely, the following assertion holds.

Theorem 3. Suppose that Ω is a connected piecewise $C^{1,\delta}$ - domain with $\delta \in (0\,,\,1)$ such that for any Γ_i the emissivity ε is constant and the operator K_i satisfies $\|\mathrm{K}_i\| < 1$. If $\tilde{f} \in \mathrm{X}'$, then the problem (4.1) has a solution.

Proof: (see [4] for details) From Lemma 5 we have that the non-local term is coercive in L^5 . Now we do not have the compactness arising from the trace theorem. Instead of that we have to use the compactness of K_i . This can be done by introducing a fixed point problem $F(q) = q$ in $L^{5/4}(\Gamma)$ where $F(q) = h(\mathrm{T}q)\big|_\Gamma$ and $\mathrm{T}q$ is the solution of the problem

$$a_1(\mathrm{T}q,\omega) + a_2(\mathrm{T}q,\omega) + \sum_i \int_{\Gamma_i} \varepsilon\, h(\mathrm{T}q)\,\omega = \langle \tilde{f},\omega \rangle + \sum_i \int_{\Gamma_i} \varepsilon \mathrm{K}_i \left(1-(1-\varepsilon)\,\mathrm{K}_i\right)^{-1} \varepsilon\, q\, \omega \quad .$$

In fact one can show that there exists a ball in $L^{5/4}(\Gamma)$ that is mapped into itself by F . Moreover, F is compact as K_i 's are compact. Hence by Schauder's fixed point argument [3 , 7], there exists a fixed point for F . The corresponding $\mathrm{T}q$ is a solution of our problem.

5.2 Temperature dependent conductivity

The results of section 4 can be generalized to the case when the coefficient of heat conductivity k is a function of the temperature T. Introducing the Kirchhoff transform

$$\mathrm{K}(\mathrm{T}) = \int_{\mathrm{T}_0}^{\mathrm{T}} k(z)\, dz \tag{5.1}$$

we have

$$k(\mathrm{T})\, \nabla \mathrm{T} = \nabla \mathrm{K}(\mathrm{T}) \tag{5.2}$$

Since $k(\mathrm{T})$ is assumed to be strictly positive for all T then the transformation is invertible and we can write $\mathrm{T} = \mathrm{K}(\mathrm{T})$. Furthermore, the mapping $\mathrm{K} \to \mathrm{T}$ is monotone. Hence, writing the problem in terms of K instead of T we arrive to the abstract formulation (4.1) with the exception that $h(\mathrm{K}) = \sigma |\mathrm{T}(\mathrm{K})|^3\, \mathrm{T}(\mathrm{K})$ instead of $\sigma\, |\mathrm{T}|^3\, |\mathrm{T}|$.

5.3 Temperature dependent emissivity

In some applications the emissive properties of materials depend strongly on temperature. Thus it is vital to consider some cases when ε is a function of temperature. Assume that ε satisfies the following conditions

(i) $\varepsilon(\bullet)$ is continuous as a function of temperature.

(ii) $\varepsilon(\mathrm{T})\, h(\mathrm{T})$ is increasing as a function of T on $\Gamma \,\Upsilon\, \Gamma_2$.

(iii) There exists ε_0 , ε_1 such that $0 < \varepsilon_0 \le \varepsilon(\mathrm{T}) \le \varepsilon_1 \le 1 \quad \forall\, \mathrm{T}$.

Theorem 4. Let Ω be a connected domain which satisfies condition (i) or (ii) of Lemma 6 . Suppose that $\tilde{f} \in L^\infty(\Omega)$, the intensity q^∞ is bounded, and that $\varepsilon = \varepsilon(\mathrm{T})$ satisfies the above three conditions. Then there exists a solution for (4.1).

Proof: We can construct sub-and supersolutions φ and ψ in $L^\infty(\Gamma)$ that are simultaneously valid for all $\varepsilon \in L^\infty(\Gamma)$, $\varepsilon_0 \le \varepsilon \le \varepsilon_1$. Let [T] denote the truncated boundary temperature [T] = min $(\psi\,, \max(\varphi\,, \mathrm{T}))$. Clearly [T] $\in L^\infty(\Gamma)$ $\forall\, \mathrm{T} \in \mathrm{H}^1(\Omega)$. We now choose $\mathrm{T}_0 \in \mathrm{H}^1(\Omega)$ arbitrarily and solve T_1 from

$$a_1(\mathrm{T}_1\,,\omega) + a_2^{\mathrm{T}_1}(\mathrm{T}_1\,,\omega) + \sum_i a_i^{\mathrm{T}_0}\left([\mathrm{T}_1],\omega\right) = <\tilde{f}\,,\omega> \quad \forall\,\omega \tag{5.3}$$

Where a^{T} denotes a with $\varepsilon = \varepsilon(\mathrm{T})$.

The problem (5.3) has sub-and supersolutions φ and ψ . Hence, there exists a solution. Furthermore, if we choose $\omega = \mathrm{T}_1$ we obtain

$$\alpha \,\|\mathrm{T}_1\|_{\mathrm{H}^1}^2 \le a_1(\mathrm{T}_1\,,\mathrm{T}_1) + a_2^{\mathrm{T}_1}(\mathrm{T}_1\,,\mathrm{T}_1) \le \left\|\tilde{f}\right\|_{(\mathrm{H}^1)'} \|\mathrm{T}_1\|_{\mathrm{H}^1} + c(\varphi\,,\psi)\,\|\mathrm{T}_1\|_{\mathrm{H}^1}\,. \tag{5.4}$$

In similar way we can construct T_{j+1} when T_j is known. The sequence $\{\mathrm{T}_j\}$ is bounded in $\mathrm{H}^1(\Omega)$ I $L^5(\Gamma_2)$. Hence it has a subsequence denoted by $\{\mathrm{T}_j\}$ that converges weakly to T^* in $\mathrm{H}^1(\Omega)$ I $L^5(\Gamma_2)$. Thus, in particular, $a_1(\mathrm{T}_j\,,\omega) \to a_1(\mathrm{T}^*\,,\omega)$. On the boundary Γ we have from the trace theorem that $\mathrm{T}_j \to \mathrm{T}^*$ in $L^2(\Gamma)$. But from this we deduce that T_j converges pointwise almost everywhere. On the other hand both $\varepsilon(\mathrm{T}_j)$ and $[\mathrm{T}_j]$ are uniformaly bounded and converge also pointwise almost everywhere. Hence they converge strongly to $\varepsilon(\mathrm{T}^*)$ and $[\mathrm{T}^*]$ respectively in $L^p(\Gamma)$ for any $p < \infty$. Now we can claim that $a_i^{\mathrm{T}_{j-1}}\left([\mathrm{T}_j],\omega\right)$ tends to $a_i^{\mathrm{T}^*}\left([\mathrm{T}^*],\omega\right)$. Using equation (2.11), we can write

$$a_i^{\mathrm{T}_{j-1}}\left([\mathrm{T}_j],\omega\right) = \left((\mathrm{I} - \mathrm{K}_i)\,q_i\,,\omega\right),$$

where q_j is the solution of

$$\left(\left(\mathrm{I} - \left(1 - \varepsilon\left(\mathrm{T}_{j-1}\right)\right)\mathrm{K}_i\right)q_j\,,\tilde{\omega}\right) = \left(\varepsilon\left(\mathrm{T}_{j-1}\right)h\left([\mathrm{T}_j]\right),\tilde{\omega}\right), \quad \forall\,\tilde{\omega} \in L^5(\Gamma_i)\,.$$

Since $\|\mathrm{K}_i\| \le 1$ we observe that

$$\|q_j\|_{L^p} \le \frac{\varepsilon_1}{\varepsilon_0}\,\|h\,[\mathrm{T}_j]\,\|_{L^p} \le c\;.$$

Hence a subsequence converges weakly to q^* q^* in L^p . Rewriting the above equation as

$$\left((\mathrm{I} - \mathrm{K}_i)\,q_j\,,\tilde{\omega}\right) = \left(\varepsilon\left(\mathrm{T}_{j-1}\right)h\left([\mathrm{T}_j]\right),\tilde{\omega}\right) - \left(\varepsilon\left(\mathrm{T}_{j-1}\right)\mathrm{K}_i\,q_j\,,\tilde{\omega}\right)$$

we notice that the convergence of the left hand side and the first term on the right hand side is clear. For the last term we first deduce from the compactness of K_i that $\mathrm{K}_i\,q_j$ converges strongly in L^p . As $\varepsilon(\mathrm{T}_j)$ converges strongly in L^5 for any $s < \infty$ we conclude that q^* is the solution of the problem

$$\left(\left(\mathrm{I} - \left(1 - \varepsilon(\mathrm{T}^*)\right)\mathrm{K}_i\right)q^*\,,\tilde{\omega}\right) = \left(\varepsilon\left(\mathrm{T}^*\right)h\left([\mathrm{T}^*]\right),\tilde{\omega}\right),$$

which proves the convergence of a_i 's. Finally, we can write

$$a_2^{\mathrm{T}_j}\left(\mathrm{T}_j\,,\omega\right) = -a_1\left(\mathrm{T}_j\,,\omega\right) - \sum_i a_i^{\mathrm{T}_{j-1}}\left([\mathrm{T}_j],\omega\right) + <\tilde{f}\,,\omega> \; = \; <F_j\,,\omega>.$$

Now , F_j converges strongly to F^* in $\left(H^1(\Omega) \cap L^5(\Gamma_2)\right)'$. This and monotonicity of a_2 imply that $a_2^{T_j}\left(T_j\,,\omega\right) \to a_2^{T^*}\left(T^*\,,\omega\right)$ and consequently, that T^* is a solution of

$$a_1\left(T^*\,,\omega\right) + a_2^{T^*}\left(T^*\,,\omega\right) + \sum_i a_i^{T^*}\left([T^*]\,,\omega\right) = < \widetilde{f}\,,\omega >.$$

As φ and ψ are sub-and supersolutions for any ε, $\varepsilon_0 \le \varepsilon \le \varepsilon_1$, in particular for $\varepsilon = \varepsilon\left(T^*\right)$ we have $[T^*] = T^*$ and therefore, T^* is a solution of the problem (4.1) with temperature dependent emissivity.

5.4 Time dependent case

Here we discuss briefly the extension of the above results to the parabolic case. Using the notation (4.1) we can write the parabolic problem as

$$\int_\Omega T_t\, u + \widetilde{a}\left(T\,,u\right) = < f\,,u >_{X'\times X} \quad ,\ \forall\, u \in X\,, \quad a.e. \quad t \in [0\,,\tau] \tag{5.5}$$

with initial condition $T(0) = T_0 \in L^2(\Omega)$. In the absence of the non-linear terms on Γ_2 and Γ_i the normal space for the solution would be

$$Y = L^2\left(0\,,\tau : H^1(\Omega) \cap H^1\left(0\,,\tau : \left(H^1(\Omega)\right)'\right)\right).$$

However, in three-dimensional case the non-linear term is well defined for functions in Y .

In order to be able to work with standard spaces we will implicitly assume some additional regularity for the solutions. Namely, we suppose that there exists sub-and supersolutions $\varphi \le \psi$, $\varphi\,,\psi \in Y \cap L^2\left([0\,,\tau]\times\left(\Gamma_2 \cup \Gamma_i\right)\right)$. Then we can define a truncation by setting $[T] = \min\left(\max\left(T\,,\varphi\right),\psi\right)$.

Theorem 5. Suppose that $f \in L^2\left(0\,,\tau : \left(H^1(\Omega)\right)'\right)$ and $T_0 \in L^2(\Omega)$ are such that the problem (5.5) has sub-and supersolutions $\varphi \le \psi$ with properties $\varphi(0) \le T_0 \le \psi(0)$ and $\varphi\,,\psi \in Y \cap L^\infty\left([0\,,\tau]\times\left(\Gamma_2 \cup \Gamma_i\right)\right)$. Then if Ω satisfies the assumption of theorem 1, the problem (5.5) has a solution $T \in Y$.

Proof: We can proceed as in the proof of theorem 1. Namely, we denote $T_1 = \psi$ and define T_n by

$$< T_{nt}\,,v >_{\left(H^1\right)'\times H^1} + a_0\left(T_n\,,v\right) + a_2\left([T_n]\,,v\right) + b\left([T_n]\,,v\right) = c\left(T_{n-1}\,,v\right) \quad a.e.\ t \in [0\,,\tau] \tag{5.6}$$

for all $v \in H^1(\Omega)$ and with initial condition $T_n(0) = T_0$.

The above problem is monotone and coercive. Hence it has a unique solution in Y . As the proof of theorem 1, we can conclude that if T_{n-1} is a supersolution, then $T_n \le T_{n-1}$ and T_n is also a supersolution. Also the sequence T_n is bounded from below by φ . Hence there exists a limit which is a solution of our problem.

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