International Journal of Modern Physics A
Vol. 21, Nos. 19 & 20 (2006) 3989–4002

A SYSTEMATIC STUDY ON NONRELATIVISTIC QUARKONIUM INTERACTION

SAMEER M. IKHDAIR
Department of Physics, Near East University,
Nicosia, North Cyprus, Turkey
sameer@neu.edu.tr

RAMAZAN SEVER
Department of Physics, Middle East Technical University,
06531 Ankara, Turkey
sever@metu.edu.tr

Received 12 August 2005
Revised 24 October 2005

A recently proposed strictly phenomenological static quark–antiquark potential belonging to the generality $V(r) = -Ar^{-\alpha} + \kappa r^{\beta} + V_0$ is tested with heavy quarkonia in the context of the shifted large N-expansion method. This nonrelativistic potential model fits the spin-averaged mass spectra of the $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ quarkonia within a few MeV and also the five experimentally known leptonic decay widths of the $c\bar{c}$ and $b\bar{b}$ vector states. Further, we compute the hyperfine splittings of the bottomonium spectrum as well as the fine and hyperfine splittings of the charmonium spectrum. We give predictions for not yet observed B_c splittings. The model is then used to predict the masses of the remaining quarkonia and the leptonic decay widths of the two pseudoscalar $c\bar{b}$ states. Our results are compared with other models to gauge the reliability of the predictions and point out differences.

Keywords: Bound state energy; decay widths; shifted large N-expansion method; quarkonia; mesons.

PACS numbers: 12.39.Pn, 14.40-n, 12.39.Jh, 14.65.-q, 13.30.Gd

1. Introduction

The charm-beauty (B_c) quarkonium states provide a unique window into heavy quark dynamics. The properties of the B_c mesons are of special interest, since they are the only quarkonia consisting of two heavy quarks with different flavors and are also intermediate to the charmonium and bottomonium systems. Additionally, because they carry flavor they cannot annihilate into gluons so are more stable with widths less than a hundred keV. Excited B_c meson states lying below BD (and BD* or B*D) threshold can only undergo hadronic (pionic) or radiative transitions to the ground pseudoscalar state which then decays weakly. This results in a rich

spectroscopy of narrow radial and orbital excitations which are considerably more stable than their charmonium and bottomonium analogues. The discovery of the B_c meson by the Collider Detector at Fermilab (CDF) Collaboration[1] in $p\bar{p}$ collisions at $\sqrt{s} = 1.8$ TeV with an observed pseudoscalar mass $M_{B_c}(1\text{S}) = 6.40\pm0.39\pm0.13$ GeV has inspired new theoretical interest in the study of the B_c spectroscopy in the framework of heavy quarkonium theory.[2–15]

On the other hand, for bottomonium system, the ALEPH collaboration in 2002 has searched for the pseudoscalar bottomonium meson, η_b, in two-photon interactions at LEP2 with an integrated luminosity of 699 pb^{-1} collected at e^+e^- center-of-mass energies from 181 GeV to 209 GeV. One candidate event is found in the six-charged-particle final state and none in the four-charged-particle final state. The candidate $\eta_b(1\text{S})$ ($\eta_b \to K_S K^- \pi^+ \pi^- \pi^+$) has reconstructed invariant mass of $9.30 \pm 0.02 \pm 0.02$ GeV.[16] Theoretical estimates (from perturbative QCD and lattice nonrelativistic QCD) to the hyperfine mass splitting for the 1S bottomonium state $\Delta_{\text{HF}}(1\text{S})$ are reported (cf. Ref. 16 and references therein).

Moreover, the inconsistent Crystal Ball measurement for the pseudoscalar charmonium mass, $M_{\eta_c(2\text{S})} = 3594 \pm 5$ MeV[17] has been given for more than 20 years, before a new measurement of $3654 \pm 6 \pm 8$ MeV was available by Belle Collaboration in summer of 2002[18] for the exclusive $B \to K K_S K^- \pi^+$ decays. It is close to the $\eta_c(2\text{S})$ mass observed by the same group in the experiment $e^+e^- \to J/\psi \eta_c$ where $M_{\eta_c(2\text{S})} = 3622 \pm 12$ MeV was found.[19] It is giving rise to a small hyperfine splitting for the 2S charmonium state, $\Delta_{\text{HF}}(2\text{S}, \text{exp}) = M_{\psi(2\text{S})} - M_{\eta_c(2\text{S})} = 32 \pm 14$ MeV.[20]

Very recently, several more new measurements have been appeared that support the Belle value: BaBar measures $3630.8\pm3.4\pm1.0$ MeV[21] (an earlier analysis resulted in $3632.2 \pm 5.0 \pm 1.8$ MeV[22]), CLEO II gives $3642.7 \pm 4.1 \pm 4.0$ MeV[23] (CLEO III prelim: $3642.5 \pm 3.6\pm$? MeV) and a different Belle analysis yields 3630 ± 8 MeV.[24] The $\Delta_{\text{HF}}(2\text{S})$ splitting calculated from a naive average of the central values of all measurements except Crystal Ball is 47 MeV. Further, the following mass values have been obtained: $M_{J/\psi(1\text{S})} = 3096.917 \pm 0.010 \pm 0.007$ MeV and $M_{\psi(2\text{S})} = 3686.111 \pm 0.025 \pm 0.009$ MeV. The relative measurement accuracy reached 4×10^{-6} for the $J/\psi(1\text{S})$, 7×10^{-6} for the $\psi(2\text{S})$ and is approximately 3 times better than that of the previous precise experiments in Refs. 25 and 26. The new result for the mass difference is $M_{\psi(2\text{S})} - M_{J/\psi(1\text{S})} = 589.194 \pm 0.027 \pm 0.011$ MeV.[2]

Consequently, such observations of hyperfine splittings for the 2S charmonium and 1S bottomonium spectra have inspired new theoretical interest in the study of the hyperfine splittings of the charmonium and bottomonium states as well as their spectra.[5,14,27–29] Badalian and Bakker[27] calculated the hyperfine splitting for the 2S charmonium state, $\Delta_{\text{HF}}(2\text{S}, \text{theory}) = 57 \pm 8$ MeV, in their recent work. Recksiegel and Sumino developed a new formalism[28] based on improved perturbative QCD approach to compute the fine and hyperfine splittings of charmonium and bottomonium.[28,29]

As a result, one is able to test the validity of the conventional phenomenological potential models where the core potential follow from simple ansatzes by

comparing our theoretical predictions of the spectrum of heavy quarkonia in terms of agreement with the experimental data with respect to the estimated uncertainties. Thus, these phenomenological potential models are not *a priori* connected to a fundamental QCD parameters.[4–8] In general, where the masses of bottomonium and charmonium states have been measured, the experimental uncertainties are much smaller than the theoretical uncertainties.[20] The only two exceptions are the very poorly measured masses of $\eta_b(1\mathrm{S})$ and $\eta_c(2\mathrm{S})$. With suitable potential model, very good agreement with the observed spectra can be obtained for the charmonium and bottomonium states (e.g. Ref. 15). These studies established the nonrelativistic nature of the heavy quarkonium systems and, in overall, a unified shape of the interquark potential in the distance region $0.5 \lesssim r \lesssim 5\ \mathrm{GeV}^{-1}$.

In this work we extend our previous analysis of the shifted large-N expansion technique (SLNET) developed for the Schrödinger wave equation[4,30–32] and then applied to semirelativistic and relativistic wave equations[5,32,33] to reproduce the hyperfine splittings of the $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ spectra using a recently proposed phenomenological potential.[15] The calculations of the bottomonium hyperfine splittings constitute predictions of the yet unobserved states. The motivation of the present work is to give a detailed analysis of mass spectra and decay widths for $c\bar{c}$, $b\bar{b}$ and B_c systems using a recently proposed phenomenological potential form.[15] We also calculate the masses of the recently found new charmonium $\eta_c(2\mathrm{S})$ and the searched bottomonium $\eta_b(1\mathrm{S})$ mesons together with the hyperfine splittings of their states.

The outline of this paper is as follows. In Sec. 2, we first review briefly the analytic solution of the Schrödinger equation for unequal mass case ($m_{q_i} \neq m_{q_j}$). Section 3 is devoted for the spin-averaged quarkonium masses of spin triplet states. The leptonic decay widths are briefly reviewed in Sec. 4. Finally, Sec. 5 contains our conclusions.

2. Schrödinger Mass Spectrum

We limit our discussion to the following generality of potentials:[5,15,34]

$$V(r) = -Ar^{-\alpha} + \kappa r^{\beta} + V_0\,, \tag{1}$$

where A, κ, α and β are nonnegative constants whereas V_0 is taking any sign. These static quarkonium potentials are monotone nondecreasing, and concave functions satisfying the condition[4,5,30,32,33]

$$V'(r) > 0 \text{ and } V''(r) \leq 0\,. \tag{2}$$

At least ten potentials of this generality, but with various values of the parameters, have been proposed in the literature. Cornell potential has $\alpha = \beta = 1$, Lichtenberg potential has $\alpha = \beta = 0.75$, Song–Lin potential has $\alpha = \beta = 0.5$, and the logarithmic potential of Quigg and Rosner corresponds to $\alpha = \beta \to 0$, have recently been studied (cf. e.g. Refs. 4–6, 30, 32, 33 and references therein). The Song's potential

used in Ref. 34 has $\alpha = \beta = 2/3$. Potentials with $\alpha \neq \beta$ have also been popular. Thus, Martin potential $\alpha = 0$, $\beta = 0.1$,[4–6,30,32,33] while Grant, Rosner and Rynes[35] prefer $\alpha = 0.045$, $\beta = 0$. Heikkilä, Törnquist and Ono[36] tried $\alpha = 1$, $\beta = 2/3$. More successful potential known in literature as Indiana potential[37] and the Richardson potential.[38] Recently, Motyka and Zalewski[15] have also explored the quality of fit in the region $0 \leq \alpha \leq 1.2$, $0 \leq \beta \leq 1.1$ of the α, β plane. The good fittings of Ref. 15 for some potential models with experiment provides that potential (1) is very reasonable with coordinates $\alpha = 1$, $\beta = 0.5$.

In the present work, in order to get a good fit, we test the second generality (1) with $\alpha = 1$, $\beta = 0.5$. Therefore, the nonrelativistic phenomenological potential used by Motyka and Zelawiski[15] for the $q_i\bar{q}_i$ and $q_i\bar{q}_j$ systems has the form

$$V(r) = -\frac{0.325250}{r} + 0.70638\sqrt{r} - 0.78891\,, \tag{3}$$

where $V(r)$, $\sqrt{r}$ and r^{-1} are all in units of GeV. The potential model (3) is convincing as it approaches to the perturbative QCD formula in the short-distance region and approached to the confining potential in the long-distance region. Consequently, in short-distance region, this potential involves the r^{-1} Coulombic dependence corresponding to one gluon exchange which is approaching to the perturbative QCD formula. The expected part of the potential which is linear in r, as in Cornell potential,[39] is not seen. Such a linearly rising potential as in Ref. 39 which is capable of confining quarks permanently can give rise to spectrum of particles containing light quarks is rough accord with experiment.[40] Probably the $b\bar{b}$ and $c\bar{b}$ are too small to reach sufficiently far into the asymptotic region of linear confinement. On the other hand, the charmonia are two large to reach sufficiently far into the confining potential in the large distance particularly for excited states near and above the open flavor threshold. Perhaps a more flexible potential would exhibit the linear part, but one may be observing an effect of the expected screening of the interaction between the heavy quarks by the light sea quarks.[41] The corresponding potential (3) is very reasonable (cf. e.g. Ref. 15).

The quark masses are

$$m_c = 1.3959 \text{ GeV}\,, \quad m_b = 4.8030 \text{ GeV}\,. \tag{4}$$

For the $c\bar{b}$ quarkonium we use the reduced mass

$$\mu_{cb} = \frac{m_c m_b}{m_c + m_b} = 1.0816 \text{ GeV}\,. \tag{5}$$

We follow Ref. 15 in choosing the centers of gravity of the triplets for the practical reasons that the masses of the spin singlets for the $b\bar{b}$ quarkonium are not known or very poorly measured.

For two particles system, we shall consider the N-dimensional space Schrödinger equation for any spherically symmetric central potential $V(r)$. If $\psi(\mathbf{r})$ denotes

the Schrödinger's wave function, a separation of variables $\psi(\mathbf{r}) = Y_{l,m}(\theta,\phi)u(r)/r^{(N-1)/2}$ gives the following radial Schrödinger equation ($\hbar = c = 1$)[4,6,30–32]

$$\left\{-\frac{1}{4\mu}\frac{d^2}{dr^2} + \frac{[\bar{k}-(1-a)][\bar{k}-(3-a)]}{16\mu r^2} + V(r)\right\} u(r) = E_{n,l}u(r)\,, \tag{6}$$

with $\mu = \frac{m_{q_i}m_{q_j}}{m_{q_i}+m_{q_j}}$ is the reduced mass for the two quarkonium composite particles. Here, $E_{n,l}$ denotes the Schrödinger binding energy of meson, and $\bar{k} = N + 2l - a$, with a representing a proper shift to be calculated later on and l is the angular quantum number. We follow the shifted $1/N$ or $1/\bar{k}$ expansion method[4,5,30–33] by defining

$$V(x(r_0)) = \sum_{m=0}^{\infty}\left(\frac{d^m V(r_0)}{dr_0^m}\right)\frac{(r_0 x)^m}{m!Q}\bar{k}^{(4-m)/2}\,, \tag{7}$$

and also the energy eigenvalue expansion[4,5,30,32,33]

$$E_{n,l} = \sum_{m=0}^{\infty}\frac{\bar{k}^{(2-m)}}{Q}E_m\,, \tag{8}$$

where $x = \bar{k}^{1/2}(r/r_0 - 1)$ with r_0 is an arbitrary point where the Taylor's expansions is being performed about and Q is a scale parameter to be set equal to $\bar{k}^2$ at the end of our calculations. Following the approach presented by Refs. 4, 30 and 32, we give the necessary expressions for calculating the binding energies:

$$E_0 = V(r_0) + \frac{Q}{16\mu r_0^2}\,, \tag{9}$$

$$E_1 = \frac{Q}{r_0^2}\left[\left(n_r + \frac{1}{2}\right)\omega - \frac{(2-a)}{8\mu}\right]\,, \tag{10}$$

$$E_2 = \frac{Q}{r_0^2}\left[\frac{(1-a)(3-a)}{16\mu} + \alpha^{(1)}\right]\,, \tag{11}$$

$$E_3 = \frac{Q}{r_0^2}\alpha^{(2)}\,, \tag{12}$$

where $\alpha^{(1)}$ and $\alpha^{(2)}$ are two useful expressions given by Imbo *et al.*[31] and also the scale parameter Q is defined by the relation

$$Q = 8\mu r_0^3 V'(r_0)\,. \tag{13}$$

Thus, for the $N = 3$ physical space, the Schrödinger binding energy to the third order is[4,30]

$$E_{n,l} = V(r_0) + \frac{1}{2}r_0 V'(r_0) + \frac{1}{r_0^2}\left[\frac{(1-a)(3-a)}{16\mu} + \alpha^{(1)} + \frac{\alpha^{(2)}}{\bar{k}} + O\left(\frac{1}{\bar{k}^2}\right)\right]\,, \tag{14}$$

where the shifting parameter, a, is defined by

$$a = 2 - (2n_r + 1)\left[3 + \frac{r_0 V''(r_0)}{V'(r_0)}\right]^{1/2}, \tag{15}$$

and the root, r_0, is being determined via

$$1 + 2l + (2n_r + 1)\left[3 + \frac{r_0 V''(r_0)}{V'(r_0)}\right]^{1/2} = [8\mu r_0^3 V'(r_0)]^{1/2}, \tag{16}$$

with $n_r = n - 1$ is the radial quantum number and n is the principal quantum number. Once r_0 is found via Eq. (16), then the Schrödinger binding energy of the $q_i\bar{q}_j$ system in (14) becomes relatively simple and straightforward. Hence, the bound state mass of the $q_i\bar{q}_j$ system is written as

$$M(q_i\bar{q}_j)_{nl} = m_{q_i} + m_{q_j} + 2E_{n,l}, \tag{17}$$

where m_{q_i} and m_{q_j} are the masses of the quark and antiquark, respectively. The expansion parameter $1/N$ or $1/\bar{k}$ becomes smaller as l becomes larger since the parameter $\bar{k}$ is proportional to n which it appears in the denominator in higher-order correction.

3. Spin-Averaged Masses of Spin Triplet States

Since the systems that we investigate in the present work are often considered as nonrelativistic system, then our treatment is based upon Schrödinger equation with a Hamiltonian μ

$$H_o = -\frac{\nabla^2}{2\mu} + V(r) + V_{SS}, \tag{18}$$

where V_{SS} is the spin–spin part. Recently, the spin–spin part, in momentum space ($q = \mu$), was found to be[42]

$$V_{SS}(m_1, m_2, q) = \frac{\mathbf{s}_1 \cdot \mathbf{s}_2}{3m_1 m_2} g_s^2(q)\left[\frac{N_c^2 - 1}{N_c} c_3(q, m_1) c_3(q, m_2) - 6N_c d(q)\right], \tag{19}$$

where N_c is the number of colors, $g_s(q)$ is the running coupling constant, $c_3(q, m) = \left(\frac{\alpha_s(q)}{\alpha_s(m)}\right)^{-9/25}$ is Wilson coefficient[42,43] and $d(q) = \frac{N_c^2-1}{8N_c^2}\left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)}\right)^{-9/25}\left[1 - \left(\frac{\alpha_s(q)}{\alpha_s(m_2)}\right)^{-18/25}\right]$. The formula (19) improves upon the one-loop perturbative calculation in two important respects: (i) it is independent of μ and (ii) it includes the higher order logarithmic terms.

If the coefficients are calculated at tree level; i.e. $c_3(\mu, m) = 1$, $d(\mu) = 0$, the potential reduces to the Eichten–Feinberg result.[44] And if these coefficients are expanded to order $\alpha_s(\mu)$ then it reduce to the one-loop spin–spin part in the nonrelativistic case[45] which is responsible for the hyperfine splitting of the mass levels[7,8,46–49]

$$V_{SS} \to V_{\rm HF} = \frac{32\pi\alpha_s}{9m_{q_i}m_{\bar{q}_j}}\left(\mathbf{s}_1 \cdot \mathbf{s}_2 - \frac{1}{4}\right)\delta^3(\mathbf{r}), \tag{20}$$

adapted from the Breit–Fermi Hamiltonian. The number $\frac{1}{4}$ substituted from the product of the spins corresponds to the recent assumption that the unperturbed nonrelativistic Hamiltonian gives the energy of the triplet states. Since for the states with orbital angular momentum $L > 0$ the wave function vanishes at the origin, the shift effect only the S states. Thus, the only first order effect of this perturbation is to shift the ${}^1S_0$ states down in energy by

$$\Delta E_{\mathrm{HF}} = \frac{32\pi\alpha_s}{9m_{q_i}m_{\bar{q}_j}}|\psi(0)|^2\,, \tag{21}$$

with the wave function at the origin is calculated by using the expectation value of the potential derivative via[4–6,30,49,50]

$$|\psi(0)|^2 = \frac{\mu}{2\pi}\left\langle\frac{dV(r)}{dr}\right\rangle\,. \tag{22}$$

In order to apply the last formula one needs the value of the wave function at the origin — this is obtained by solving the Schrödinger equation with the nonrelativistic Hamiltonian and the coupling constant. In this approach, the QCD strong coupling constant $\alpha_s(4\mu^2)$, on the renormalization point μ^2 is not an independent parameter. It can be connected (in the $\overline{MS}$ renormalization scheme) through the two-loop relation[10,15]

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0}\frac{\left(\frac{2\mu}{\Lambda_{\overline{MS}}^{(n_f)}}\right)^2 - 1}{\ln\left[\left(\frac{2\mu}{\Lambda_{\overline{MS}}^{(n_f)}}\right)^2\right]}\,, \tag{23}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$. Like most authors (cf. e.g. Refs. 4–8, 15), the strong coupling constant $\alpha_s(m_c^2)$ is fitted to the experimental charmonium hyperfine splitting number $\Delta_{\mathrm{HF}}(1\mathrm{S},\mathrm{exp}) = 117 \pm 2$ MeV[4,20] yields

$$\alpha_s(m_c^2) = 0.254\,. \tag{24}$$

Knowing the coupling at the scale m_c^2 we obtain the couplings at other scales as follows. The number of flavors (n_f) is put equal to three for $4\mu^2 \leq m_c^2$ (we are not interested in the region $4\mu^2 \leq m_s^2$), equal to four for $m_b^2 \geq 4\mu^2 \geq m_c^2$ and equal to five for $4\mu^2 \geq m_b^2$ (we are not interested in the region $4\mu^2 \geq m_t^2$). Then the value of $\alpha_s(4\mu^2 = m_c^2)$ from (23) is used to calculate $\Lambda_{\overline{MS}}^{(n_f=3)}$ and $\Lambda_{\overline{MS}}^{(n_f=4)}$. Using the known value of $\Lambda_{\overline{MS}}^{(n_f=4)}$ and the formula form (23) we find the value

$$\alpha_s(m_b^2) = 0.200\,. \tag{25}$$

From this the value of $\Lambda_{\overline{MS}}^{(n_f=5)}$ is found. Note that this supports our model since a different choice of the Hamiltonian would in general lead to a different value of the wave function at the origin and to a different determination of $\alpha_s(m_c^2)$ from the same hyperfine splitting. Then the estimate of $\alpha_s(m_Z^2)$ would, of course, be

also different. For the hyperfine splitting of the $c\bar{b}$ quarkonium we use the coupling constant

$$\alpha_s(4\mu_{cb}^2) = 0.224\,, \tag{26}$$

so that in each case the scale is twice the reduced mass of the quark–antiquark system.

The calculated quarkonium masses together with hyperfine splittings are given in Tables 1–3. The hyperfine mass splittings of $c\bar{c}$ and $b\bar{b}$ predicted by the potential

Table 1. Schrödinger bound-state mass spectrum of $c\bar{c}$ quarkonium (in MeV). ΔX denotes the difference between the mass of particle X and the center of gravity of the spin triplet part of the multiplet, where X belongs.

State	EQ[7]	GJ[51]	MZ[15]	This work	Expt.[20]
1^3S_1	3097	3097	3097	3097	3096.87 ± 0.04
$\Delta 1^1S_0$	−117	−117	−117	−117	−117
$1P$(c.o.g)	3492	3526	3521	3521	3525.3 ± 0.2
2^3S_1	3686	3685	3690	3694	3685.96 ± 0.09
$\Delta 2^1S_0$	−78	−68	−72	−65.9	−92/−32
$1D$(c.o.g)				3806	(1^3D_1 state) 3769.9 ± 25
$2P$(c.o.g)				3944	
3^3S_1				4078	4040 ± 10
$2D$(c.o.g)				4150	4159 ± 20?
$3P$(c.o.g)				4265	
4^3S_1				4437	4415 ± 6
$3D$(c.o.g)				4430	
5^3S_1				4628	

Table 2. Schrödinger bound-state mass spectrum of $b\bar{b}$ quarkonium (in MeV). ΔX denotes the difference between the mass of particle X and the center of gravity of the spin triplet part of the multiplet, where X belongs.

State	EQ[7]	KR[11]	MZ[15]	This work	Expt.[20]
1^3S_1	9464		9460	9460	9460
$\Delta 1^1S_0$	−87		−56.7	−57.9	(160)
$1P$(c.o.g)	9873	9903	9900	9900	9900
2^3S_1	10007		10023	10031	10023
$\Delta 2^1S_0$	−44		−28	−23.2	
$1D$(c.o.g)	10127	10156	10155	10155	
$2P$(c.o.g)	10231	10259	10260	10261	10260
3^3S_1	10339	—	10355	10364	10355
$2D$(c.o.g)	—	10441	10438	10438	
$3P$(c.o.g)	—	10520	10525	10527	
4^3S_1				10614	
$3D$(c.o.g)				10666	
5^3S_1				10820	

Table 3. Schrödinger bound-state mass spectrum of $c\bar{b}$ ($b\bar{c}$) quarkonium in MeV. ΔX denotes the difference between the mass of particle X and the center of gravity of the spin triplet part of the multiplet, where X belongs.

State	MZ[15]	CK[59]	EQ[7]	R[60]	G[12]	GJ[51]	This work
1^3S_1	6349	6355	6337	6320	6317	6308	6349
$\Delta 1^1S_0$	−58	−45	−73	−65	−64	−41	−58.2
$1P$(c.o.g)	6769	6764	6736	6753	6728	6753	6769
2^3S_1	6921	6917	6899	6900	6902	6886	6926
$\Delta 2^1S_0$	−33	−27	−43		−35	−33	−30
$1D$(c.o.g)			—				7040
$2P$(c.o.g)	7165	7160	7160		7122		7165
3^3S_1							7288
$2D$(c.o.g)							7359
$3P$(c.o.g)							7464
4^3S_1							7567
$3D$(c.o.g)							7619
5^3S_1							7800

Table 4. Level splittings in charmonium and bottomonium (in MeV).

Level splitting	Expt.	Ref. 7[a]	Ref. 15[a]	Ref. 14[a]	Ref. 61[a]	Ref. 56[b]	Ref. 57[b]	Ref. 29[c]	This work
$\Delta_{\rm HF}^{(c\bar{c})}(2S)$ $= M_{\psi(2S)} - M_{\eta_c(2S)}$	92/32	78	72	98	92	43	—	38	66
$\Delta_{\rm HF}^{(b\bar{b})}(1S)$ $= M_{\Upsilon(1S)} - M_{\eta_b(1S)}$	(160)	87	57	60	45	—	51	44	58
$\Delta_{\rm HF}^{(b\bar{b})}(2S)$ $= M_{\Upsilon(2S)} - M_{\eta_b(2S)}$	—	44	28	30	28	—	—	21	23

[a]Potential model.
[b]Lattice.
[c]Perturbative QCD.

model is listed with some other models in Table 4. Therefore, as for the hyperfine splittings in Table 4, all of the potential model calculations try to reproduce the old experimental values, while the lattice calculations and perturbative QCD favor the new values. No confirmed experimental data to check these predictions are available yet. Let us note, however, that the unconfirmed experimental splitting of the $2S(c\bar{c})$ level $-92/32$ MeV — is much bigger/lower than expected from the potential models. In all cases, where comparison with the other models are significantly smaller than the splittings found by Eichten and Quigg[7] and similar to, but usually a little smaller than, the splittings calculated by Gupta and Johnson.[51]

One can also try to compare our results with more ambitious approaches. A careful analysis in the framework of QCD sum rules[52] finds the hyperfine splitting of the bottomonium $\Delta_{\rm HF}(1{\rm S}, {\rm theory}) = 63^{+29}_{-51}$ MeV. The central value agrees to several MeV very well with our expectation, but the uncertainty is too large to

distinguish between the potential models. A lattice calculation[9] gives the hyperfine splitting $\Delta_{\rm HF}(1{\rm S}, {\rm theory}) = 60$ MeV with a large uncertainty. Again the central value is close to our model, but the uncertainty is big enough to be consistent with all the potential models quoted here.

4. Leptonic Decay Widths

The leading terms in the leptonic decay widths of the heavy quarkonia are proportional to the squares of the wave functions at the origin. Therefore, they are significant only for the S states. For the $c\bar{c}$ and $b\bar{b}$ quarkonium systems, we shall consider the decays of the n^3S_1 (vector) states into pairs of charge conjugated charged leptons, e.g. for definiteness into e^+e^- pairs. For the $c\bar{b}$ quarkonium we consider the decays of the n^1S_0 (pseudoscalar) states into $\tau\nu_\tau$ pairs. Since the probability of such decays contains as a factor the square of the lepton mass, the decays into lighter leptons are much less probable.

The decay widths of the vector $c\bar{c}$ and $b\bar{b}$ quarkonium systems into charged lepton pairs are usually calculated from the QCD corrected Van Royen–Weiskopf formula[53]

$$\Gamma_{V\to\bar{l}l} = 16\pi\alpha^2 e_q^2 \frac{|\psi(0)|^2}{M_V^2}\left(1 - \frac{16\alpha_s(m_q^2)}{3\pi}\right) . \tag{27}$$

For vector mesons containing light quarks this formula leads to paradoxes (cf. Ref. 54 and references therein). For quarkonia, however, the main problem seems to be the QCD correction. Thus, in order to get quantitative predictions it is necessary to include higher order corrections which are not known. In order to estimate the missing terms we tried two simple forms. Exponentialization of the first correction

$$C_1(\alpha_s(m_q^2)) = \exp\left(-\frac{16\alpha_s(m_q^2)}{3\pi}\right) , \tag{28}$$

and Padeization

$$C_2(\alpha_s(m_q^2)) = \frac{1}{1 + \frac{16\alpha_s(m_q^2)}{3\pi}} . \tag{29}$$

We use the average of these two estimates as our estimate of the QCD correction factor extended to higher orders. The difference between C_1 and C_2 is our crude evaluation of the uncertainty of this estimate. The resulting leptonic widths are collected in Table 5. Further, we have the relation

$$\Gamma_{V\to\bar{l}l} = \frac{9}{8}\frac{4m_q^2}{M_V^2}\frac{\alpha^2 e_q^2}{\alpha_s(m_q^2)} C_{\rm av}\Delta E_{\rm HF} , \tag{30}$$

Table 5. Leptonic widths (in KeV).

State	EQ[7]	MZ[15]	This work	Expt.
$1^3S_1(c\bar{c})$	8	4.5 ± 0.5	6.72 ± 0.49	5.3 ± 0.4
$2^3S_1(c\bar{c})$	3.7	1.9 ± 0.2	2.66 ± 0.19	2.1 ± 0.2
$1^3S_1(b\bar{b})$	1.7	1.36 ± 0.07	1.45 ± 0.07	1.32 ± 0.05
$2^3S_1(b\bar{b})$	0.8	0.59 ± 0.03	0.52 ± 0.02	0.52 ± 0.03
$3^3S_1(b\bar{b})$	0.6	0.40 ± 0.02	0.35 ± 0.02	0.48 ± 0.08
$1^1S_0(c\bar{b})$	4×10^{-8}	2.8×10^{-8}	3.58×10^{-8}	—
$2^1S_0(c\bar{b})$	—	1.6×10^{-8}	1.89×10^{-8}	—

where $C_{\rm av}$ is the averaged QCD correction factor. With our choice of parameters this formula reduces to

$$\Gamma_{V\to\bar{l}l} = F(q)\frac{4m_q^2}{M_V^2}\Delta E_{\rm HF}\,, \tag{31}$$

with $F(c) = 7.07 \times 10^{-5}$ and $F(b) = 2.43 \times 10^{-5}$.

The formula for the leptonic widths of the pseudoscalar $c\bar{b}$ quarkonium reads

$$\Gamma_{\tau\nu_\tau} = \frac{G^2}{8\pi} f_{B_c}^2 |V_{cb}|^2 M_{B_c} m_\tau^2 \left(1 - \frac{m_\tau^2}{M_{B_c}^2}\right)^2\,, \tag{32}$$

where G is the Fermi constant, $V_{cb} \approx 0.04$ is the element of the Cabibbo–Kobayashi–Masakawa matrix and the decay constant f_{B_c} is given by the formula (cf. e.g. Ref. 55)

$$f_{B_c}^2 = \frac{12|\psi(0)|^2}{M_{B_c}^2}\bar{C}^2(\alpha_s)\,, \tag{33}$$

where $\bar{C}(\alpha_s)$ is QCD correction factor. Formally this decay constant is defined in terms of the element of the axial weak current

$$\langle 0|A_\mu(0)|B_c(q)\rangle = i f_{B_c} V_{cb} q_\mu\,. \tag{34}$$

The QCD correction factor is[51]

$$\bar{C}(\alpha_s) = 1 - \frac{\alpha_s(\mu_{cb}^2)}{\pi}\left[2 - \frac{m_b - m_c}{m_b + m_c}\ln\frac{m_b}{m_c}\right]\,. \tag{35}$$

With our parameters $\bar{C}(\alpha_s) \approx 0.905$ and since this is rather close to unity, we use it without trying to estimate the higher order terms.

Substituting the numbers one finds the decay widths given in Table 5. The corresponding decay constants for the ground state and for the first excited S-state of the $c\bar{b}$ quarkonium are found to be $f_{B_c} = 492$ MeV and $f_{B_c} = 338$ MeV (cf. e.g. Ref. 6).

Let us note the convenient relation

$$f_{B_c}^2 = \frac{27\mu_{cb}}{8\pi\alpha_s(4\mu_{cb}^2)}\frac{m_b + m_c}{M_{B_c}}\bar{C}^2(\alpha_s)\Delta E_{\rm HF}\,, \tag{36}$$

which for our values of the parameters yields

$$f_{B_c} = 65.2\sqrt{\frac{6199}{M_{B_c}}}\sqrt{\Delta E_{\rm HF}}\,, \tag{37}$$

where all the parameters are in suitable powers of MeV.

5. Conclusions

Figure 1 shows the behavior of the present potential form in comparison with some other potential forms considered in Ref. 4. From this figure it is noticed that, all potential forms are nearly the same at large r, but the behavior seems to be different at very small r ($r < 0.5$ GeV^{-1}, about 0.1 fm), due to the variation of the one-gluon exchange term in each potential model. Further, the present potential model containing six free parameters: the three parameters in the strictly nonrelativistic phenomenological potential (3), the masses of the c and b quarks (4) and the strong coupling constant at the m_c scale (23). This nonrelativistic single-channel model is applicable to all heavy quarkonia below their strong decay thresholds. However, the coupled-channel theory is necessary especially for the states near and above the open flavor threshold. In this regard, we give an approximate predictions to these energy levels in Tables 1–3; namely, 4S and 5S states.

Consequently, we obtain the $c\bar{c}$, $b\bar{b}$ and $c\bar{b}(b\bar{c})$ quarkonium mass spectra and also their leptonic decay widths in a approximate close agreement with up-to-date experimental findings. We also give prediction for the $c\bar{b}$ quarkonium masses and also for

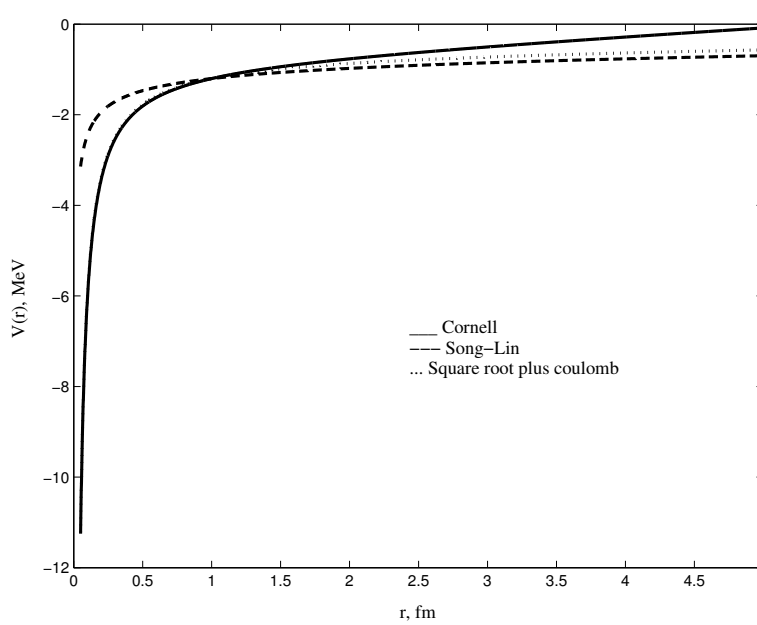

Fig. 1. The behavior of the different potential models versus r.

the leptonic widths of the pseudoscalar $c\bar{b}$ quarkonium. Our model predicts nearly an approximate hyperfine splitting for the 1S bottomonium and 2S charmonium as in the other potential models,[7,15] lattice[56,57] and perturbation QCD.[28,29,58] Further, the fine splitting in charmonium is found to be $M_{\psi(1S)} - M_{J/\psi(1S)} = 597$ MeV.

Finally, in general, the potential models seem to reproduce the experimental values much better. This feature would be understandable, since the potential models contain much more input parameters than the lattice or perturbative QCD models.

Acknowledgments

This research was partially supported by the Scientific and Technological Research Council of Turkey. The authors wish to thank S. Recksiegel for information on recent QCD-based formalism. S. M. Ikhdair would like to thank his wife, Oyoun, and his son, Musbah, for their assistance, patience and love.

References

1. CDF Collab. (F. Abe *et al.*), *Phys. Rev. D* **58**, 112004 (1998).
2. N. Brambilla *et al.*, hep-ph/0412158.
3. S. Godfrey, *Phys. Rev. D* **70**, 054017 (2004).
4. S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. A* **18**, 4215 (2003).
5. S. M. Ikhdair and R. Sever, *Int. J. Mod. Phys. A* **19**, 1771 (2004); *ibid.* **20**, 4035 (2005); *ibid.* **20**, 6509 (2005).
6. S. M. Ikhdair, hep-ph/0504107; *ibid.* hep-ph/0504176.
7. E. J. Eichten and C. Quigg, *Phys. Rev. D* **49**, 5845 (1994).
8. L. P. Fulcher, Z. Chen and K. C. Yeong, *Phys. Rev. D* **47**, 4122 (1993); *ibid.* **50**, 447 (1994); L. P. Fulcher, *Phys. Rev. D* **60**, 074006 (1999); *ibid.* **44**, 2079 (1991).
9. C. T. H. Davies *et al.*, *Phys. Lett. B* **382**, 131 (1996).
10. M. Baldicchi and G. M. Prosperi, *Phys. Lett. B* **436**, 145 (1998); *Phys. Rev. D* **62**, 114024 (2000); *Fiz. B* **8**, 251 (1999).
11. W. Kwong and J. Rosner, *Phys. Rev. D* **44**, 212 (1991); *ibid.* **38**, 279 (1988).
12. S. Gershtein *et al.*, *Phys. Rev. D* **51**, 3613 (1995).
13. N. Brambilla and A. Vairo, *Phys. Rev. D* **62**, 094019 (2000).
14. D. Ebert, R. N. Faustov and V. O. Galkin, *Phys. Rev. D* **67**, 014027 (2003); *Mod. Phys. Lett. A* **17**, 803 (2002); hep-ph/0210381; hep-ph/0110190; hep-ph/0006186.
15. L. Motyka and K. Zalewiski, *Eur. Phys. J. C* **4**, 107 (1998); *Z. Phys. C* **69**, 343 (1996).
16. ALEPH Collab. (A. Heister *et al.*), *Phys. Lett. B* **530**, 56 (2002).
17. Crystal Ball Collab. (E. Edwards *et al.*), *Phys. Rev. Lett.* **48**, 70 (1982).
18. Belle Collab. (S.-K. Choi *et al.*), *Phys. Rev. Lett.* **89**, 102001 (2002) [Erratum, *ibid.* **89**, 129901 (2002)].
19. Belle Collab. (K. Abe *et al.*), *Phys. Rev. Lett.* **89**, 142001 (2002).
20. Particle Data Group Collab. (K. Hagiwara *et al.*), *Phys. Rev. D* **66**, 010001 (2002).
21. BaBar Collab. (B. Aubert *et al.*), hep-ex/0311038.
22. BaBar Collab. (G. Wagner), hep-ex/0305083.
23. CLEO Collab. (J. Ernst *et al.*), hep-ex/0306060.
24. Belle Collab. (K. Abe *et al.*), Belle-Conf-0331.
25. A. A. Zholents *et al.*, *Phys. Lett. B* **96**, 214 (1980).

26. E760 Collab. (T. A. Armstrong *et al.*), *Phys. Rev. D* **47**, 772 (1993).
27. A. M. Badalian and B. L. G. Bakker, *Phys. Rev. D* **67**, 071901 (2003); *ibid.* **62**, 094031 (2000).
28. S. Recksiegel and Y. Sumino, *Phys. Rev. D* **67**, 014004 (2003).
29. S. Recksiegel and Y. Sumino, *Phys. Lett. B* **578**, 369 (2004).
30. S. M. Ikhdair and R. Sever, hep-ph/0507209 (to appear in *Int. J. Mod. Phys.*).
31. T. Imbo *et al.*, *Phys. Rev. D* **29**, 1669 (1984); H. Christiansen *et al.*, *Phys. Rev. A* **40**, 1760 (1989); T. D. Imbo and U. P. Sukhatme, *Phys. Rev. Lett.* **54**, 2184 (1985).
32. S. M. Ikhdair, O. Mustafa and R. Sever, *Turkish J. Phys.* **16**, 510 (1992); *ibid.* **17**, 474 (1993); S. M. Ikhdair and R. Sever, *Z. Phys. D* **28**, 1 (1993).
33. S. M. Ikhdair and R. Sever, *Z. Phys. C* **56**, 155 (1992); *ibid.* **58**, 153 (1993); *Hadronic J.* **15**, 389 (1992).
34. X. T. Song, *J. Phys. G* **17**, 49 (1991).
35. A. K. Grant, J. L. Rosner and E. Rynes, *Phys. Rev. D* **47**, 1981 (1993).
36. K. Heikkilä, N. A. Törnquist and S. Ono, *Phys. Rev. D* **29**, 110 (1984).
37. G. Fogelmann *et al.*, *Nuovo Cimento* **26**, 369 (1979).
38. J. L. Richardson, *Phys. Lett. B* **82**, 272 (1979).
39. E. Eichten *et al.*, *Phys. Rev. D* **17**, 3090 (1978); *ibid.* **21**, 203 (1980).
40. R. L. Jaffe and K. Johnson, *Comments Nucl. Part. Phys.* **7**, 107 (1977); Y. Nambu, *Phys. Rev. D* **10**, 4262 (1974).
41. K. D. Born *et al.*, *Phys. Lett. B* **329**, 325 (1994).
42. Y.-Q. Chen, Y.-P. Kuang and R. J. Oakes, *Phys. Rev. D* **52**, 264 (1995); Y.-Q. Chen and R. J. Oakes, *Phys. Rev. D* **53**, 5051 (1996).
43. A. Billoire *et al.*, *Nucl. Phys. B (Proc. Suppl.)* **4**, 199 (1988); A. F. Falk, B. Grinstein and M. E. Luke, *Nucl. Phys. B* **357**, 185 (1991).
44. E. Eichten and F. Feinberg, *Phys. Rev. Lett.* **43**, 1205 (1979); *Phys. Rev. D* **23**, 2724 (1982).
45. J. Pantaleone, S.-H. H. Tye and Y. J. Ng, *Phys. Rev. D* **33**, 777 (1986); S. N. Gupta and S. F. Radford, *Phys. Rev. D* **24**, 2309 (1981); S. N. Gupta, S. F. Radford and W. W. Repko, *Phys. Rev. D* **26**, 3305 (1982).
46. H. J. Schnitzer, *Phys. Rev. D* **13**, 74 (1975).
47. M. R. Arafah *et al.*, *Ann. Phys.* **220**, 55 (1992).
48. W. Lucha *et al.*, *Phys. Rev. D* **46**, 1088 (1992); *ibid.* **45**, 385 (1992); *ibid.* **45**, 1233 (1992); W. Lucha, F. F. Schöberl and D. Gromes, *Phys. Rep.* **200**, 127 (1991).
49. C. Quigg and J. L. Rosner, *Phys. Lett. B* **71**, 153 (1977); *Phys. Rep.* **56**, 167 (1979).
50. L. I. Abou-Salem, *Int. J. Mod. Phys. A* **20**, 4113 (2005).
51. S. N. Gupta and J. M. Johnson, *Phys. Rev. D* **54**, 2075 (1996).
52. S. Narison, *Phys. Lett. B* **387**, 162 (1996).
53. R. Van Royen and V. Weisskopf, *Nuovo Cimento A* **50**, 617 (1967); *ibid.* **51**, 583 (1967).
54. S. Narison and K. Zalewski, *Phys. Lett. B* **320**, 369 (1994).
55. E. Braaten and S. Fleming, *Phys. Rev. D* **52**, 181 (1995).
56. CP-PACS Collab. (M. Okamoto *et al.*), *Phys. Rev. D* **65**, 094508 (2002).
57. X. Liao and T. Manke, *Phys. Rev. D* **65**, 074508 (2002).
58. J. Pantaleone and S.-H. Tye, *Phys. Rev. D* **37**, 3337 (1998).
59. Y.-Q. Chen and Y.-P. Kuang, *Phys. Rev. D* **46**, 1165 (1992).
60. R. Roncaglia *et al.*, *Phys. Rev. D* **52**, 1627 (1995).
61. L. Haysak *et al.*, hep-ph/0301188.