A NOTE ON INCOMPLETE EXPONENTIAL FUNCTIONS

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1. INTRODUCTION

The recent paper by Chaudhry and Qadir [1] proposed the incomplete exponential functions in analogy to the incomplete gamma functions. The functions are defined by

$$e\left((x,t);\alpha\right) = \sum_{n=0}^{\infty} \frac{\gamma(\alpha+n,x)t^n}{\Gamma(\alpha+n)n!} \qquad (1)$$

and

$$E\left((x,t);\alpha\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n,x)t^n}{\Gamma(\alpha+n)n!}, \qquad (2)$$

where $\gamma(\cdot,\cdot)$ and $\Gamma(\cdot,\cdot)$ are the incomplete gamma functions defined by

$$\gamma(a,x) = \int_0^x t^{a-1}\exp\left(-t\right)dt$$

and

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\exp\left(-t\right)dt,$$

respectively. The paper also demonstrated an application of the incomplete exponential functions to the non–central chi–square distribution.

We would like to point out that the functions given by (1) and (2) are directly related to a known function. In fact, it is easy to see that

$$e\left((x,t);\alpha\right) = \exp(t)\left[1 - Q_\alpha\left(\sqrt{2t},\sqrt{2x}\right)\right] \qquad (3)$$

and

$$E\left((x,t);\alpha\right) = \exp(t)Q_\alpha\left(\sqrt{2t},\sqrt{2x}\right), \qquad (4)$$

where $Q(\cdot,\cdot)$ is the well known Marcum Q function [2] defined by

$$Q_M\left(\gamma,\beta\right) = \gamma^{1-M}\int_\beta^\infty u^M \exp\left\{-\frac{u^2+\gamma^2}{2}\right\} I_{M-1}(\gamma u)du,$$

In the light of the relationships (3)–(4), it is worth noting that (1)–(2) provide convergent series expansions for the Marcum Q function for small values of t.

The analytical as well as the computational properties of the Marcum Q function have been studied extensively, especially in the digital communication literature. See [3]–[5], Appendix A of [6], pages 394–395 and 411 of [7], [8]–[12], Chapter 4 of [13], and [14]–[16].

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Paper Received 30 April 2006; Revised 18 June 2006; Accepted a Technical Note 20 December 2006

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