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ORIGINAL ARTICLE

A charged spinless particle in scalar–vector harmonic oscillators with uniform magnetic and Aharonov–Bohm flux fields

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Abstract The two-dimensional solution of the spinless Klein–Gordon (KG) equation for scalar–vector harmonic oscillator potentials with and without the presence of constant perpendicular magnetic and Aharonov–Bohm (AB) flux fields is studied within the asymptotic function analysis and Nikiforov–Uvarov (NU) method. The exact energy eigenvalues and normalized wave functions are analytically obtained in terms of potential parameters, magnetic field strength, AB flux field and magnetic quantum number. The results obtained by using different Larmor frequencies are compared with the results in the absence of both magnetic field ($\omega_L = 0$) and AB flux field ($\xi = 0$) case. Effects of external fields on the non-relativistic energy eigenvalues and wave functions solutions are also precisely presented.
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1. Introduction

It is well known that the exact solution of the Schrödinger equation (SE) and relativistic wave equations for some physical potentials are very important in many fields of physics and chemistry since they contain all the necessary information for the quantum system under investigation. The hydrogen atom and the harmonic oscillator are usually given in textbooks as two of several exactly solvable problems in both classical and quantum physics (Greiner and Müller, 1994). The exact l-state solutions of the SE are possible only for a few potentials and hence approximation methods are used to obtain their solutions. According to the Schrödinger formulation of quantum mechanics, a total wave function provides implicitly all relevant information about the behavior of a physical system. Hence, if it is exactly solvable for a given potential, the wave function can describe such a system completely. Until now, many efforts have been made to solve the stationary SE with anharmonic potentials in two-dimensions (2D), three-dimensions (3D) and D-dimensional space (Ikhdair and Sever, 2008a; Dong, 2001a,b; Dong and Ma, 1998; Child et al., 2000;

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Dong, 2000) with many applications to molecular and chemical physics. The study of the SE with these potentials provides us with insight into the physical problem under consideration. However, the study of SE with some of these potentials in the arbitrary dimensions D is presented in (cf. (Dong, 2002) and the references therein). Furthermore, the study of the bound state processes is also fundamental to understanding of molecular spectrum of a diatomic molecule in quantum mechanics (Flügge, 1994). Recently, some authors have studied the bound state solutions of the l-wave Schrö dinger, Klein–Gordon (KG) and Dirac equations with some typical potentials in the presence of an equal scalar potential $S(r)$ and a vector potential $V(r)$. These potentials include the harmonic oscillator potential (Akcay and Tezcan, 2009; Ikhdair, 2012), ring-shaped Kratzer-type potential (Qiang, 2004), pseudo-harmonic oscillator potential (Ikhdair and Sever, 2007), double ring-shaped harmonic oscillator potential (Lu et al., 2005), ring-shaped pseudo-harmonic oscillator potential (Ikhdair and Sever, 2008b; Ikhdair and Sever, 2008c; Ikhdair and Sever, 2009), ring-shaped potential(Falaye, 2012a), spherically asymmetrical singular oscillator (Falaye, 2012b), Eckart potential (Falaye, 2012c), etc.

It is well known that non-relativistic quantum mechanics is an approximate theory of the relativistic one. When a particle moves in a strong potential field, the relativistic effect must be considered, which gives the corrections for non-relativistic quantum mechanics (Wang and Wong, 1988). So the motion of spin-0 and spin-1/2 particles satisfies the KG and the Dirac equations, respectively.

We shall discuss the spin-0 KG solution for the harmonic oscillator in 2D space in external magnetic field and Aharanov–Bohm (AB) flux field (Khordad, 2010; Khordad, 2011; Çetin, 2008) since the conserved quantities of the 2D harmonic oscillator generate the Lie group SU(2) (Wybourne, 1974). In the KG and the Dirac systems, Hamiltonians with equal scalar and vector harmonic oscillator potential has the same dynamical symmetries as their non-relativistic counterparts (Ginocchio, 2005; Lisboa et al., 2004; Zhang et al., 2009; Zhang and Chen, 2009). Hence, these discussions suggest that there should be a coordinate transformation connecting relativistic systems with SU(3) dynamical symmetries.

Recently, the spectral properties in a 2D charged particle (electron or hole) confined by a harmonic oscillator in the presence of an external strong uniform magnetic field $\vec{B}$ along the z direction and Aharonov–Bohm (AB) flux field created by a solenoid have been studied. The Schrödinger equation is solved exactly for its bound states (energy spectrum and wave functions) (Khordad, 2010; Khordad, 2011; Çetin, 2008). So, it is natural that the relativistic effects for a charged particle under the action of this potential could become important, especially for a strong coupling.

Recently, the 2D solution of Schrödinger equation for the Kratzer potential with and without the presence of a constant magnetic field has been investigated (Aygun et al., 2012) within the framework of the asymptotic iteration method. The energy eigenvalues are obtained analytically (numerically) for the absence (presence) of magnetic field case. The results obtained by using different Larmor frequencies ($\omega_L \neq 0$) and potential parameters are compared with the results in the absence of magnetic field case ($\omega_L = 0$). The spectral properties of an electron confined by 2D harmonic and pseudoharmonic oscillators have been studied in the presence of external fields (Ikhdair et al., 2012; Ikhdair and Hamzavi, 2012a) in the framework of the Nikiforov–Uvarov (NU) method. Very recently, we have studied the scalar charged particle exposed to relativistic scalar–vector Killingbeck potentials, i.e., harmonic oscillator potential plus Cornell potential, in the presence of magnetic and Aharonov–Bohm flux fields and obtained its energy eigenvalues and wave functions using the analytical exact iteration method (Ikhdair, 2013; Rajabi and Hamzavi, 2013).

The aim of the present work is to investigate the KG equation in 2D for an equal mixture of scalar–vector harmonic oscillator potentials in the presence and absence of constant uniform magnetic and AB flux fields that point in the z-direction. The exact bound state energy eigenvalues and normalized wave functions are calculated in the framework of the NU method (Nikiforov and Uvarov, 1988; Tezcan and Sever, 2009; Ikhdair, 2009). The non-relativistic energy eigenvalues and wave functions of our solution are presented by making an appropriate mapping of parameters. Further, special cases of KG for equal mixture of scalar–vector harmonic oscillator potentials are also presented in the presence ($\omega_L \neq 0$, $\xi \neq 0$) and absence ($\omega_L = 0, \xi = 0$) uniform fields.

The structure of this paper is as follows. We study the effect of external uniform magnetic and AB flux fields on a relativistic spinless particle (anti-particle) under equal mixture of scalar and vector harmonic oscillator potentials in Section 2. We discuss some special cases in Section 3. Finally, we give our concluding remarks in Section 4.

2. Relativistic bound states of the HO in constant external fields

The KG equation of a charged particle moving in constant magnetic and AB flux fields can be written as (Greiner, 2000; Alhaidari et al., 2006)

$$\left[c^2\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2-(E-V(r))^2+(Mc^2+S(r))^2\right]\psi(r,\phi)=0, \tag{1}$$

where the vector potential in the symmetric guage is defined by $\vec{A} = \vec{A}_1 + \vec{A}_2$ such that $\vec{\nabla} \times \vec{A}_1 = \vec{B}$ and $\vec{\nabla} \times \vec{A}_2 = 0$, where $\vec{B} = B\hat{z}$ is the applied magnetic field and $\vec{A}_2$ describes the additional Aharonov–Bohm (AB) flux field Φ_{AB} created by a solenoid in cylindrical coordinates (Bogachek and Landman, 1995; Ferkous and Bounames, 2004). The vector potential have the following azimuthal components (Çetin, 2008)

$$\vec{A}_1 = \frac{1}{2}\vec{B}\times\vec{r}, \quad \vec{A}_2 = \frac{\Phi_{AB}}{2\pi r}\hat{\phi}, \quad \vec{A} = \left(\frac{Br}{2}+\frac{\Phi_{AB}}{2\pi r}\right)\hat{\phi}. \tag{2}$$

We use the following wave function

$$\psi(r,\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}g(r), \quad m = 0, \pm1, \pm2, \ldots, \tag{3}$$

where m is the eigenvalue of angular momentum. The relationship between the attractive scalar and repulsive vector potentials is given by $S(r) = \beta V(r)$, where $-1 \leqslant \beta \leqslant 1$ is the arbitrary constant and the KG equation could be reduced to a Schrödinger-type second order differential equation as

$$\left[c^2\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2+2(EV(r)+Mc^2S(r))+S^2(r)-V^2(r)+M^2c^4-E^2\right]\psi(r,\phi) = 0, \tag{4}$$

where $V(r)$ is taken as the harmonic oscillator in the form (Qiang, 2004; Lu et al., 2005):

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$$V(r) = V_{\mathrm{HO}}(r) = \frac{1}{2}kr^2, \tag{5}$$

where $k = M\omega^2$ is the elastic coefficient (Qiang, 2004; Lu et al., 2005). Now we will treat the bound-state solutions of the two cases in Eq. (4) as follows.

2.1. The positive energy case

The positive energy states corresponding to $S(r) = +V(r)$ (i.e., $\beta = 1$ case) in the non-relativistic limit are solutions of the wave equation:

$$\left\{\frac{1}{2\mu}\left[\vec{p} + \frac{e}{c}\left(\frac{Br}{2} + \frac{\Phi_{AB}}{2\pi r}\right)\hat{\phi}\right]^2 + 2V(r) - E\right\}\psi(r,\phi) = 0, \tag{6}$$

where $\psi(r,\phi)$ stands for non-relativistic wave function. This is the Schrödinger equation for potential $2V(r)$. Thus, the choice $S(r) = +V(r)$ produces a non-relativistic limit with potential function $2V(r)$ and not $V(r)$. Accordingly, it would be natural to scale the potential term in Eq. (4) and Eq. (6) so that in the non-relativistic limit the interaction potential becomes $V(r)$ not $2V(r)$. Thus, we need to recast Eq. (4) for the $S(r) = V(r)$ as (Greiner, 2000; Xu et al., 2010; Ikhdaie and Sever, 2012b)

$$\left[c^2\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 + 2(E + Mc^2)V(r)\right]\psi(r,\phi) = (E^2 - M^2c^4)\psi(r,\phi), \tag{7}$$

and in order to simplify Eq. (7) we introduce new parameters $\lambda_1 = E + Mc^2$ and $\lambda_2 = E - Mc^2$ so that it can be reduced to the form

$$\left[c^2\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 - \lambda_1(\lambda_2 - V(r))\right]\psi(r,\phi) = 0. \tag{8}$$

Now, letting $g(r) = \frac{1}{\sqrt{r}}R(r)$ and inserting Eqs. (2), (3) and (5) into the KG Eq. (8), we obtain

$$\frac{d^2R(r)}{dr^2} + \frac{\lambda_1}{\hbar^2c^2}[\lambda_2 - U_{\mathrm{eff}}(r,\omega_L,\xi)]R(r) = 0, \tag{9}$$

with

$$U_{\mathrm{eff}}(r,\omega_L,\xi) = V_{\mathrm{HO}}(r) + \frac{M^2c^2}{\lambda_1}\omega_L^2r^2 + \frac{\hbar^2c^2}{\lambda_1}\frac{(m'^2 - 1/4)}{r^2} + \frac{2\hbar\omega_L Mc^2m'}{\lambda_1}, \tag{10a}$$

$$\omega_L = \frac{\Omega}{2},\quad \Omega = \frac{|e|B}{Mc}, m' = m + \xi,\quad \xi = \frac{\Phi_{AB}}{\Phi_0},\quad m' = 1,2,\ldots, \tag{10b}$$

where the effective potential depending on the magnitudes of two fields strength with ω_L and m' are the Larmor frequency and a new eigenvalue of angular momentum (magnetic quantum number), respectively. It is worthy to mention that the frequency Ω is called the cyclotron frequency. This is the frequency of rotation corresponding to the classical motion of a charged particle in a uniform magnetic field and $\Omega/2$ is the Larmor frequency in units of Hz (s^{-1}) (Liboff, 2003). Moreover, we take ξ as integer with the flux quantum $\Phi_0 = hc/e$. Here $V_{\mathrm{HO}}(r)$ is a pure harmonic oscillator, the second term is the harmonic oscillator-type potential and other terms are the rotational potential creating the rotational energy levels. Eq. (9) can be alternatively expressed as

$$g''(r) + \frac{1}{r}g'(r) + \left(v^2 - \gamma^2r^2 - \frac{m'^2}{r^2}\right)g(r) = 0, \tag{11}$$

with

$$v^2 = \frac{1}{\hbar^2c^2}(\lambda_1\lambda_2 - 2\hbar Mc^2\omega_L m'),\quad \gamma^2 = \frac{1}{\hbar^2c^2}\left(\frac{k\lambda_1}{2} + M^2c^2\omega_L^2\right), \tag{12}$$

where the asymptotic behaviors $g(r = 0) = 0$ and $g(r \to \infty)$ being finite. Moreover, introducing a change of variable $s = r^2$, that maps $r \in (0,\infty)$ to $s \in (0,\infty)$, we obtain second-order differential equation satisfying the radial wave function $g(s)$,

$$g''(s) + \frac{1}{s}g'(s) + \frac{1}{4s^2}(-\gamma^2s^2 + v^2s - m'^2)g(s) = 0, \tag{13}$$

Now using the basic ideas of the NU method (Nikiforov and Uvarov, 1988; Tezcan and Sever, 2009; Ikhdair, 2009), we thus obtain the energy equation:

$$v^2 = 2(1 + 2n + m')\gamma,\quad n = 0,1,2,\ldots. \tag{14}$$

with the constant parameters used in our calculations are displayed in Table 1. Inserting the values of v^2 and γgiven in Eq. (12) into Eq. (14), we arrive at the following transcendental energy formula,

$$2(2n + m' + 1)\sqrt{\left(\frac{M\omega_L}{\hbar}\right)^2 + \frac{k(E + Mc^2)}{2\hbar^2c^2}} = \frac{1}{\hbar^2c^2}[E^2 - M^2c^4 - 2\hbar Mc^2\omega_L m']. \tag{15}$$

We may find a solution to the above transcendental equation as $E = E_{KG}^{(+)}$. In the non-relativistic limit when inserting $\lambda_1 \to 2\mu$, $\lambda_2 \to E_{nm'}$ and $c = 1$ gives the desired result

$$E_{nm'}(\omega_L,\xi) = \hbar\Omega'(2n + m' + 1) + \hbar m'\omega_L,\quad n = 0,1,2,\ldots,$$
$$\Omega' = \sqrt{\omega_L^2 + \omega^2},\quad \omega = \sqrt{k/\mu}, \tag{16}$$

where the second term is the rotational energy levels. Using the NU method (Nikiforov and Uvarov, 1988; Tezcan and Sever, 2009; Ikhdair, 2009) and Table 1, we can find the radial part of the wave function (3) as

$$g(r) = C_{n,m}r^{|m'|}e^{-\gamma r^2/2}F(-n, |m'| + 1; \gamma r^2), \tag{17}$$

where the normalization constant has been calculated in Appendix A (cf. Eq. (54)). Hence, the total KG wave function (3) is obtained as follows

$$\psi_{n,m}^{(+)}(r,\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}\sqrt{\frac{2\gamma^{|m'|+1}n!}{(n + |m'|)!}}r^{|m'|}e^{-\gamma r^2/2}L_n^{(m')}(\gamma r^2), \tag{18}$$

where $L_a^{(b)}(x) = \frac{(a+b)!}{a!b!}F(-a, b+1; x)$ is the associated Laguerre polynomial and $F(-a,b;x)$ is the confluent hypergeometric function. Notice that the wave function (18) is finite and satisfying the standard asymptotic analysis (cf. Appendix A) for the limiting cases $r = 0$ and $r \to \infty$.

Table 1 Specific values of the constants in the solution of Eq. (20).

Constants	($\alpha_3 = 0$ case)
$\xi_1 = \gamma^2/4$	$\xi_2 = v^2/4$
$\xi_3 = m'^2/4$	$\alpha_1 = 1$
$\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$	$\alpha_6 = \xi_1 = \gamma^2/4$
$\alpha_7 = -\xi_2 = -v^2/4$	$\alpha_8 = \xi_3 = m'^2/4$
$\alpha_9 = \alpha_6 = \gamma^2/4$	$\alpha_{10} = m' + 1$
$\alpha_{11} = \gamma$	$\alpha_{12} = m'/2$
$\alpha_{13} = -\gamma/2$	

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As shown in Fig. 1a and Eq.(10a), the effective potential function changes in shape when the magnetic field strength increases, say; $\omega_L = 8$ and in absence of AB flux field. The energy levels are raised when the strength of the magnetic field increases and in the absence of the AB flux field $\xi = 0$. We see that the effective potential changes gradually from the pure pseudo-harmonic oscillator potential, when $\omega_L = 0$, to a pure harmonic oscillator type behavior in short potential range when $\omega_L = 8$. In Fig. 1b, the effective potential (10a) which is pseudoharmonic oscillator when $\omega_L = 0$.becomes sensitive to the increasing AB flux field $\xi = 8$ in the short range region, i.e., $0 < r < 4$ a.u.

We see from Fig. 1a, the large influence of the magnetic field on the shape of the effective potential energy (10a). It follows that when the strength of magnetic field increases, the potential becomes purely harmonic oscillator in its shape, i.e., the contribution of the centrifugal term appears for small interaction distances, $r \to 0$,

$$U_{\text{eff}}(r,\omega_L) \to \frac{1}{2}k'r^2 + \frac{d_1}{r^2} + d_2, \quad k' = k + \frac{2M^2c^2}{\lambda_1}\omega_L^2,$$
$$d_1 = \frac{\hbar^2c^2}{\lambda_1}(m^2 - 1/4), \quad d_2 = \frac{2\hbar\omega_L Mc^2 m}{\lambda_1}.$$

In Fig. 1b, the AB flux field has not much effect on the effective potential energy (10a) which is of pseudoharmonic oscillator shape.

2.2. The bound states for negative energy

When $S(r) = -V(r)$, we need to follow same procedure of the solution in the previous subsection and consider the solution given by Eq. (12) with the changes

$$\gamma^2 \to \tau^2 = \left(\frac{M\omega_L}{\hbar}\right)^2 + \frac{1}{\hbar^2c^2}\frac{k\lambda_2}{2}. \tag{19}$$

Hence, the negative energy solution for antiparticle can be readily found as

$$2(1 + 2n + m')\sqrt{\left(\frac{M\omega_L}{\hbar}\right)^2 + \frac{1}{\hbar^2c^2}\frac{k\lambda_2}{2}}$$
$$= \frac{1}{\hbar^2c^2}\left[\lambda_2\lambda_1 - 2\hbar Mc^2\omega_L m'\right], \quad m' = 1, 2, \ldots, \tag{20}$$

and the wave function is

$$\psi_{n,m}^{(-)}(r,\phi) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2\tau^{|m'|+1}n!}{(n+|m'|)!}}e^{im\phi}r^{|m'|}e^{-\tau r^2/2}L_n^{(m')}(\tau r^2). \tag{21}$$

The negative energy states are free fields since under these conditions Eq. (6) can be rewritten as

$$\left[-\frac{1}{2\mu}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + E\right]\psi_{n,m}(r,\phi) = 0, \tag{22}$$

which is a simple free-interaction mode. Further, the parameters given in Eqs. (12) and (19) become

$$v = \sqrt{\frac{2M}{\hbar^2}(E - \hbar\omega_L m')}, \quad \tau = \frac{M\omega_L}{\hbar}. \tag{23}$$

Thus, Eq. (16) with $k = 0$, gives the following energy formula

$$E_{nm'}^{(-)} = (2n + m' + |m'| + 1)\hbar\omega_L, \tag{24}$$

and hence the wave function reads

$$\psi_{nm'}^{(-)}(r,\phi) = \sqrt{\frac{2\left(\frac{M\omega_L}{\hbar}\right)^{m'+1}n!}{(n+m')!}}r^{m'}e^{-\frac{M\omega_L}{2\hbar c}r^2}L_n^{(m')}\left(\frac{M\omega_L}{\hbar}r^2\right)\frac{1}{\sqrt{2\pi}}e^{im\phi}. \tag{25}$$

3. Discussions

In this section we briefly study some special cases and relationship between our results and some other authors':

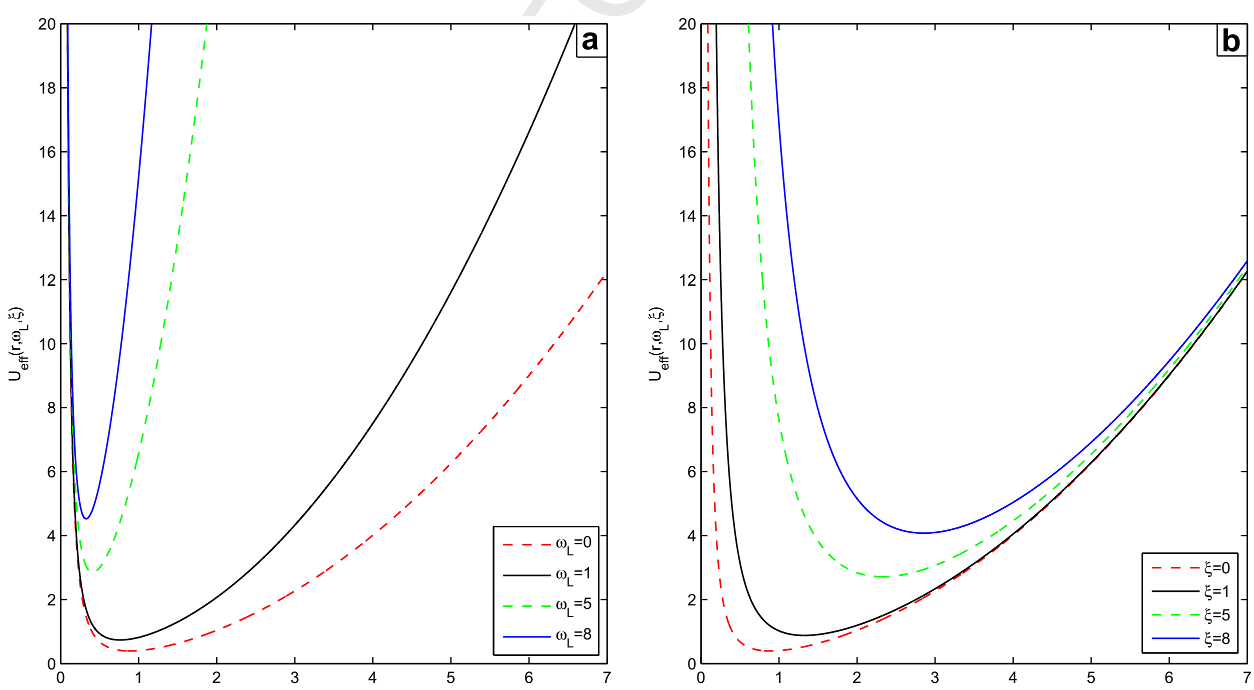

Fig. 1 (color online) The KG effective potential function for (a) $\omega_L \neq 0$, $\xi = 0$ and (b) $\omega_L = 0, \xi \neq 0$. Here $M = c = k = 1$.

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3.1. Schrödinger-harmonic oscillator under external fields

In the non-relativistic limit, the Schrödinger equation in 2D is

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}[E-U_{\mathrm{eff}}(r,\omega_L,\xi)]R(r)=0, \tag{26a}$$

$$U_{\mathrm{eff}}(r,\omega_L,\xi)=V_{\mathrm{HO}}(r)+\frac{1}{2}\omega_L^2r^2+\frac{\hbar^2}{2\mu}\frac{(m'^2-1/4)}{r^2}+\hbar\omega_L m', \tag{26b}$$

and hence the energy spectrum (16) can be rewritten simply as

$$E_{nm'}(\xi,\omega_L)=\hbar\Omega'(2n+m'+1)+\hbar\omega_L m', \tag{27}$$

and the wave function becomes

$$\psi_{n,m'}^{(+)}(r,\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}\sqrt{\frac{2b^{m'+1}n!}{(n+m')!}}r^{m'}e^{-br^2/2}L_n^{(m')}(br^2),\quad b=\frac{\mu}{\hbar}\Omega'. \tag{28}$$

In the absence of two fields (i.e., $\omega_L=0$, $\xi=0$), the 2D energy spectrum being

$$E_{nm}=\hbar\omega(2n+m+1),$$

and the wave function

$$\psi_{n,m}^{(+)}(r,\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}\sqrt{\frac{2b^{m+1}n!}{(n+m)!}}r^{m}e^{-br^2/2}L_n^{(m)}(br^2),\quad b=\frac{\mu}{\hbar}\ \omega. \tag{29}$$

In Fig. 2a, we plot the effective potential for the case of low vibrational ($n=0,1,2,3$) and rotational ($m=1$) levels for various Larmor frequencies $\omega_L=0,1,5,8$ and $\xi=0$ case. As shown in Fig. 2a, and Eq. (26b), the effective potential function changes in shape as well as the bound state energy eigenvalues increase when $\omega_L=8$. It is shown that the energy levels are raised when the strength of the magnetic field increases and in the absence of AB flux field. It is also obvious that the effective potential changes gradually from the pure pseudo-harmonic oscillator potential, which is a no-magnetic ($\omega_L=0$) and the AB flux ($\xi=0$) fields case, to a pure harmonic oscillator type behavior in a short potential range when the strength of the applied magnetic field is increased to $\omega_L=8$. If we consider a strong magnetic field case $\omega_L=8$ which has the shape of pure harmonic oscillator potential function, the energy difference between adjacent energy levels are nearly equal which is a known characteristics of the pure harmonic oscillator potential. In Fig. 2b, the effective potential (26b) is the pseudoharmonic oscillator in the absence of magnetic field $\omega_L=0$ becomes sensitive to the increasing AB flux field $\xi=8$ in the short range region for small r, i.e., $0<r<4$ a.u.

We see from Fig. 2a that if B strength increases, then effective potential energy (26b) is a harmonic oscillator in its shape, the centrifugal term $1/r^2$ is dominant for small r values:

$$U_{\mathrm{eff}}(r,\omega_L)\to\frac{1}{2}k''r^2+\frac{c_1}{r^2}+c_2,\quad k''=k+\omega_L^2,\quad c_1=\frac{\hbar^2}{2\mu}(m^2-1/4),\quad c_2=\hbar\omega_L m.$$

However, in Fig. 2b, the AB flux field has not much effect on the effective potential energy (26b) which is purely pseudoharmonic oscillator.

On the other hand, we give some numerical values to the energy states with and without external fields. In Tables 2 and 3, we show the effect of the magnetic field and AB flux field, respectively, on the low vibrational n and rotational m relativistic energy states of the harmonic oscillator potential. As shown in Table 2, when the magnetic field is not applied and without the AB flux field ($\omega_L=0,\xi=0$), the spacing between the energy levels of the effective potential is narrow and decreases with increasing n. But when the magnetic field strength increases, the energy levels of the effective potential increase and spacings between states also increase. In Table 3, when the AB flux field is applied and without magnetic field, the energy states become degenerate for various values of n and m and for various AB flux field strength values. In Tables 4 and 5, we show the effect of the magnetic field and AB flux field, respectively, on the low vibrational n and rotational m nonrelativistic energy states of the harmonic oscillator potential. As shown in Table 4, when the magnetic field is not applied and without the AB flux field ($\omega_L=0$, $\xi=0$), the energy states are equally spaced (the pure harmonic oscillator case). But when the magnetic field strength is raised, the energy levels of the effective potential increase and spacings between states also increase. In Table 5, when the AB flux field is applied and without the magnetic field, the energy states become degenerate and equally spaced for various values of n and m and for various AB flux field strength values.

A first look at Tables 2 and 4 shows that in the absence of the uniform magnetic field $B=0$ ($\omega_L=0$), the energy spacing is constant value, i.e., $\Delta E=2=$ constant, for any quantum number m value in the nonrelativistic case. However, in the relativistic case, the energy spacing ΔE decreases with the increasing of n states for $m=0$, i.e., $\Delta E=1.25696,\ 1.02758,\ 0.90721,\ldots,$ and when $m=1$, $\Delta E=1.11943,\ 0.95997,\ 0.86438.$ It is obvious that the increasing of the quantum number leads to a decrease in the energy spacing which is becoming continuous for large value of m. On the other hand, when $B>0$, the nonrelativistic energy spacing shows increment $\Delta E=2.82843=$ constant when $\omega_L=1$ Hz and $\Delta E=16.12454=$ constant when $\omega_L=8$ Hz. Thus, we see that the energy spacing increases with increasing larmor frequency but remains constant for all n and m states. Overmore, considering the relativistic solution, we notice that when $\omega_L=1$ Hz, $\Delta E=1.38537,\ 1.08598,\ 0.94318,\ldots,$ for $m=0$ and $\Delta E=1.10665,\ 0.95623,\ 0.86331,\ldots,$ for $m=1$ due to the corrections in eigenenergies. This demonstrates that energy spacing decreases with increasing m quantum number. It follows that for large m, the states become continuous. Further, when applied magnetic field increases for which $\omega_L=8$ Hz, $\Delta E=2.94449,\ 2.06549,\ 1.69472,\ldots,$ for $m=0$ and $\Delta E=2.06647,\ 1.69523,\ 1.4758,\ldots,$ for $m=1$. It is obvious that increasing magnetic field leads to an increase in the energy spacing. We conclude that increasing the quantum number m leads to a decrease in the energy spacing. From Tables 3 and 5, we can make a similar analysis for the AB flux field ξ.

3.2. KG-harmonic oscillator problem

The energy spectrum of relativistic spinless particle in the absence of magnetic and AB flux fields has the form:

$$\sqrt{2k}(1+2n+m)\hbar=\lambda_2\sqrt{\lambda_1}, \tag{30}$$

which is identical to Eq. (41) of (Lu et al., 2005) with $l'=m'-1/2$. The above energy formula can be reduced to its non-relativistic limit:

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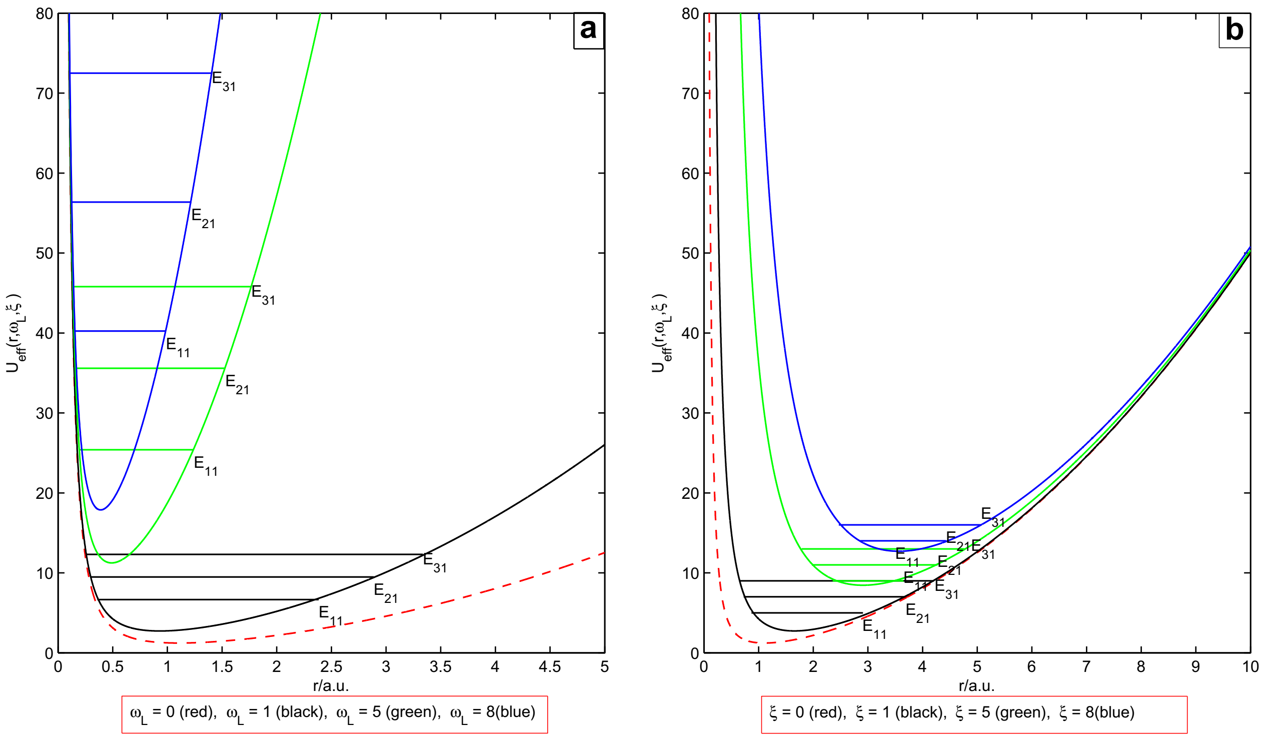

Fig. 2 The Schrödinger effective potential function and corresponding bound state energy levels (E_{nm}) in low vibrational ($n = 0, 1, 2, 3$) and rotational ($m = 1$) levels for (a) $\omega_L \neq 0$, $\xi = 0$ and (b) $\omega_L = 0, \xi \neq 0$. Here $\mu = k = 1$.

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$$E_{nm} = (1 + 2n + m)\hbar\omega_D, \quad \omega_D = \sqrt{k/M}. \tag{31}$$

The wave function can be expressed as

$$\psi^{(+)}_{n,m}(r,\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \times \sqrt{\frac{2\left(\frac{M\omega_D}{\hbar}\right)^{m+1} n!}{(n+m)!}} r^m e^{-M\omega_D r^2/2\hbar} L_n^{(m)}\left(\frac{M\omega_D}{\hbar} r^2\right), \tag{32}$$

Following (Nikiforov and Uvarov, 1988; Tezcan and Sever, 2009; Ikhdair, 2009), the energy equation of the relativistic spinless particle subject to the harmonic oscillator field is

$$n'\hbar c\sqrt{2k} - \sqrt{\lambda_1}\lambda_2 = 0, \quad n' = 1, 2, \ldots, \tag{33}$$

where $n' = 1 + |m| + 2n, n = 0, 1, 2, \ldots$, which is completely identical to Eq. (11) and Eq. (26) in (Qiang, 2004) when one uses the notation $k = 2V_0/r_0^2 = M\omega_D^2$. Following (Qiang, 2004), Eq. (33) has three solutions, the only real solution giving energy is

$$E_{nm} = \frac{1}{3}\left(Mc^2 + M^2c^4 T^{-1/3} + T^{1/3}\right), \tag{34}$$

with

$$T = 27kn'^2\hbar^2c^2 - 8M^3c^6 + 3n'\hbar c\sqrt{3k(27kn'^2\hbar^2c^2 - 16M^3c^6)}. \tag{35}$$

The wave function takes the form

$$\psi^{(+)}_{n,m}(r,\phi) = \sqrt{\frac{D^{|m|+1} n!}{\pi(n+|m|)!}} r^{|m|} e^{-Dr^2/2} L_n^{(|m|)}(Dr^2) e^{im\phi}, \quad D = \frac{M\omega_D}{\hbar}. \tag{36}$$

If one expands Eq. (33) as a series of λ_2, it becomes

$$n'\hbar = \sqrt{\frac{M}{k}}\left[\lambda_2 + \frac{1}{4Mc^2}\lambda_2^2 - \frac{1}{32M^2c^4}\lambda_2^3 + O\left(\lambda_2^4\right)\right], \tag{37}$$

and taking the first order of λ_2 by neglecting the higher order relativistic corrections, we finally arrive at the non-relativistic solution:

$$E'_{nm} = E_{nm} - Mc^2 = \hbar\sqrt{\frac{k}{M}}(1 + 2n + |m|) = (1 + |m| + 2n)\hbar\omega_D, \quad n = 0, 1, 2, \ldots, \tag{38}$$

and wave function resembles the one given in Eq. (36).

Here, we explain in detail the physical behaviors of the energy eigenvalues due to increasing strength of the external uniform magnetic field and AB flux field. We try to discuss the nonrelativistic case for clarity and simplicity. The energy levels in Eq. (16) are commonly referred to as Landau levels. We see that when $B = 0$ and $\xi = 0$, the spacing between Landau levels is the constant value

$$\Delta E = 2\hbar\omega_0, \tag{39}$$

for any m. The behaviors of energy levels of the simple harmonic oscillator are equally spaced (lowest energy of harmonic oscillator is $\hbar\omega$). Notice that in the classical limit $\hbar \to 0$, the spacing between levels ΔE goes to zero (nearly continuous). However, as $B > 0$, the spacing between levels becomes

$$\Delta E = 2\hbar\Omega' = 2\hbar\sqrt{\omega_L^2 + \omega_0^2}. \tag{40}$$

Note that there is an increment in the energy spacing. In particular, the equally spaced Landau levels corresponding to Larmor frequencies $\omega_L = \omega_0, 3\omega_0, 5\omega_0, 8\omega_0, \ldots$, become $2\sqrt{2}\hbar\omega_0, 2\sqrt{10}\hbar\omega_0, 2\sqrt{26}\hbar\omega_0, 2\sqrt{65}\hbar\omega_0, \ldots$, respectively. Thus,

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Table 2 For various Larmor frequencies ω_L and without AB flux field ($\xi = 0$), the spinless relativistic energy eigenvalues (E_{nm} in atomic units) of a particle under the harmonic oscillator potential field with $\hbar = c = M = k = 1$.

m	n	E_{nm}								
		$\omega_L = 0$	$\omega_L = 1$	$\omega_L = 2$	$\omega_L = 3$	$\omega_L = 4$	$\omega_L = 5$	$\omega_L = 6$	$\omega_L = 7$	$\omega_L = 8$
0	0	1.83929	2.04353	2.40325	2.75615	3.08137	3.38066	3.65815	3.91752	4.16166
	1	3.09625	3.4289	4.03986	4.65113	5.21808	5.74094	6.22602	6.67941	7.10609
	2	4.12383	4.51488	5.26359	6.03138	6.75146	7.4193	8.04087	8.62301	9.17158
	3	5.03104	5.45806	6.30283	7.18744	8.02542	8.80667	9.53605	10.2205	10.8663
1	0	2.50976	3.16597	3.89307	4.55265	5.14478	5.68284	6.17802	6.63858	7.0706
	1	3.62919	4.27262	5.12444	5.93706	6.68087	7.36315	7.99437	8.58337	9.13707
	2	4.58916	5.22885	6.16824	7.09549	7.95634	8.7516	9.49036	10.1815	10.8323
	3	5.45354	6.09216	7.09891	8.11893	9.0768	9.96689	10.7966	11.5745	12.3081

Table 3 For various AB flux field $\xi$and without magnetic field ($\omega_L = 0$), the spinless relativistic energy eigenvalues (E_{nm} in atomic units) of a particle under the harmonic oscillator potential field.

m	n	E_{nm}								
		$\xi = 0$	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$	$\xi = 5$	$\xi = 6$	$\xi = 7$	$\xi = 8$
0	0	1.83929	2.50976	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966
	1	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313
	2	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.63137.0	7.0	7.35892
	3	5.03104	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901	8.05108
1	0	2.50976	3.09625	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166
	1	3.62919	4.12383	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313	7.0
	2	4.58916	5.03104	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901
	3	5.45354	5.85966	6.25166	6.6313	7.0	7.35892	7.70901	8.05108	8.38582

Table 4 For various Larmor frequencies ω_L and without AB flux field ($\xi = 0$), the nonrelativistic energy eigenvalues (E_{nm} in atomic units) of a particle under the harmonic oscillator potential field with $\hbar = M = k = 1$.

m	n	E_{nm}								
		$\omega_L = 0$	$\omega_L = 1$	$\omega_L = 2$	$\omega_L = 3$	$\omega_L = 4$	$\omega_L = 5$	$\omega_L = 6$	$\omega_L = 7$	$\omega_L = 8$
0	0	1.0	1.41421	2.23607	3.16228	4.12311	5.09902	6.08276	7.07107	8.06226
	1	3.0	4.24264	6.7082	9.48683	12.3693	15.2971	18.2483	21.2132	24.1868
	2	5.0	7.07107	11.1803	15.8114	20.6155	25.4951	30.4138	35.3553	40.3113
	3	7.0	9.89949	15.6525	22.1359	28.8617	35.6931	42.5793	49.4972	56.4358
1	0	2.0	3.82843	6.47214	9.32456	12.2462	15.198	18.1655	21.1421	24.1245
	1	4.0	6.65685	10.9443	15.6491	20.4924	25.3961	30.3311	35.2843	40.249
	2	6.0	9.48528	15.4164	21.9737	28.7386	35.5941	42.4966	49.4264	56.3735
	3	8.0	12.3137	19.8885	28.2982	36.9848	45.7922	54.6621	63.5685	72.4981

Table 5 For various AB flux field ξ and without magnetic field ($\omega_L = 0$), the nonrelativistic energy eigenvalues (E_{nm} in atomic units) of a particle under the harmonic oscillator potential field.

m	n	E_{nm}								
		$\xi = 0$	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$	$\xi = 5$	$\xi = 6$	$\xi = 7$	$\xi = 8$
0	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
	1	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0
	2	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0
	3	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0
1	0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	1	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
	2	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0
	3	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0	16.0

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we find that the energy spectrum of a confined electron changes from a nearly continuous one for $B = 0$ ($\omega_L = 0$) to a discrete spectrum for $B > 0(\omega_L \neq 0)$. On the other hand, when $B = 0$ and $\xi \neq 0$, the spacing between Landau levels is the constant value as in Eq. (37). It follows that the AB flux field has no influence on the energy spacings between different states. the behavior of the relativistic harmonic oscillator in the presence of magnetic field will increase the energy spacing due to the relativistic effects as we can expect correction terms to the nonrelativistic term.

Here, we explain in detail physical behaviors of why the eigenenergies increase or decrease with the Larmor frequency. We define the quantity $e\hbar/2Mc$ as a Bohr magneton. It has the value

$$\mu_b = \frac{|e|\hbar}{2Mc} = 0.927 \times 10^{-20} \text{ erg/gauss}, \tag{41}$$

and the relationship between Bohr magneton, magnetic field and Larmor frequency is given by

$$\hbar\omega_L = \mu_b B. \tag{42}$$

For an electron, one finds the magnetic moment is directly proportional to its spin angular momentum. It is given by

$$\vec{\mu} = -\frac{e}{Mc}\vec{S} = -\frac{e\hbar}{2Mc}\vec{\sigma} = -\mu_b\vec{\sigma}. \tag{43}$$

We now consider the problem of calculating the eigenstates and eigenenergies of the present model, i.e., a spinning but otherwise fixed electron in a constant uniform magnetic field that points in the z direction. To solve this problem we use the Schrödinger equation. For the case at hand, it appears as

$$\widehat{H} = -\vec{\mu}\cdot\vec{B} = \mu_b\vec{\sigma}\cdot\vec{B} = \mu_b B\sigma_z = \hbar\omega_L\hat{\sigma}_z, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\mu} = -\mu_b\vec{\sigma}. \tag{44}$$

Setting $|\psi\rangle = \begin{pmatrix} f \\ g \end{pmatrix}$ gives

$$\widehat{H}|\psi\rangle = E|\psi\rangle \rightarrow \hbar\omega_L\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = E\begin{pmatrix} f \\ g \end{pmatrix}, \tag{45}$$

or, equivalently,

$$\hbar\omega_L f = Ef, -\hbar\omega_L g = Eg. \tag{46}$$

If $f \neq 0$, $g = 0$, then $E = +\hbar\omega_L = +\mu_b B$. If $g \neq 0$, $f = 0$, then $E = -\hbar\omega_L = -\mu_b B$. Thus we obtain the normalized eigenstates and eigenenergies

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E = +\hbar\omega_L = +\mu_b B, \tag{47a}$$

$$\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E = -\hbar\omega_L = -\mu_b B. \tag{47b}$$

In the state of higher energy, the spin of the electron is parallel to $\vec{B}$, so the magnetic moment is antiparallel to $\vec{B}$ and the interaction energy $-\vec{\mu}\cdot\vec{B}$ is maximum. In the state of lower energy, the spin of the electron is antiparallel to $\vec{B}$, so the magnetic moment is parallel to $\vec{B}$ and the interaction energy $-\vec{\mu}\cdot\vec{B}$ is minimum. Notice that the energy formulas (15) and (16) are mainly dependent on the magnetic quantum numbers $m = 0, \pm 1, \pm 2, \ldots$, which are influenced by the magnetic field pointing along z-axis which is splitting energy to maximum and minimum levels. For further details on the physical properties of similar potential models under the influence of uniform electric, magnetic and AB flux fields, one is advised to refer to other works (Ikhdair, 2012; Ikhdair et al., 2012; Ikhdair and Hamzavi, 2012b; Ikhdair and Hamzavi, 2012c).

4. Concluding remarks

To sum up, in this paper, we have studied the solutions of the KG and Schrödinger equations in two-dimensional space with the harmonic oscillator interaction for low vibrational and rotational energy levels without and with a constant magnetic field having the arbitrary Larmor frequency and AB flux field. We have used the NU method for $\omega_L \neq 0$ (with magnetic field) and $\xi \neq 0$ (with AB flux field) and obtained analytical expressions for bound state energies and wave functions of the relativistic spinless particle subject to a harmonic oscillator interaction in terms of external uniform magnetic and AB flux fields in any vibrational n and rotational m states. The above results show that the problems of relativistic quantum mechanics can be also solved exactly as in the non-relativistic ones. We considered the solution of both positive (particle) and negative (anti-particle) KG energy states. The Schrö dinger bound state solution is found as a non-relativistic limit of the present model. It is noticed that the solution with an equal mixture of scalar–vector potentials can be easily reduced into the well-known Schrö dinger solution for a particle with an interaction potential field and a free field, respectively. We have also studied the bound-state solutions for some special cases including the non-relativistic limits (Schrödinger equation for harmonic oscillator under external magnetic and AB flux fields) and the KG equation for harmonic oscillator interactions. The results show that the splitting is not constant and dependent mainly on the strength of the external magnetic field and AB flux field. In order to show the effect of constant magnetic and AB flux fields on the vibrational and rotational energy levels of the harmonic oscillator we plot the effective potential and corresponding energy levels with the increasing Larmor frequency and flux field for special potential parameters. We have seen that the effective potential function and corresponding energy levels are raised in energy when magnetic and AB flux field strengths increase. The effective potential function behavior gradually changes from the pure pseudo-harmonic oscillator to a pure harmonic oscillator shape in short potential range as the magnetic and AB flux field strengths increase.

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Appendix A. Asymptotic analysis

We consider here a more shorter solution to Eq. (11). The 2D Schrö dinger-type equation satisfying the radial wave function $R(r)$,

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$$-\frac{d^2R(r)}{dr^2}+\left(\gamma^2r^2+\frac{(m'^2-1/4)}{r^2}\right)R(r)=\nu^2R(r), \tag{48}$$

where $g(r) = r^{-1/2}R(r)$. We can solve Eq. (48) by using Eq. (17) in (Ikhdair and Sever, 2007) with the replacements: $2\mu E'_{nL}/\hbar^2 \to \nu^2$, $2\mu\ B^2/\hbar^2 \to \gamma^2$ and $2L+1 \to\ 2m'$ in Eq. (19) of (Ikhdair and Sever, 2007) to obtain our Eqs. (15) and (17). A first inspection on the asymptotic analysis of Eq. (48), we find out that if r approaches 0, the radial wave function $R(r) \sim r^p$, $p = m' + 1/2 > 0$ and if $r \to \infty$, $R(r) \sim \exp(-\gamma r^2/2)$, hence both solutions are satisfying the boundary conditions of the radial wave function $g(r = 0) = 0$ and $g(r \to \infty) \to 0$. In the entire range $r \in (0,\infty)$, we consider the general solution $g(r) = r^{m'}\exp(-\gamma r^2/2)L(r)$, $m' > 0$, where $L(r)$ is the associated Laguerre polynomials. Letting

$$R(r) = r^{m'+1/2}\exp(-\gamma r^2/2)L(r), \tag{49}$$

and substituting Eq. (49) into Eq. (48) gives

$$\frac{d^2L(r)}{dr^2}+2\left(-\gamma r+\frac{m'+1/2}{r}\right)\frac{dL(r)}{dr}+4\gamma nL(r)=0. \tag{50}$$

Introducing a new variable $z = \gamma r^2$, Eq. (50) can be rewritten as

$$z\frac{d^2L(z)}{dz^2}+(m'+1-z)\frac{dL(z)}{dz}+nL(z)=0, \tag{51}$$

which is the well known differential equation whose solution is the associated Laguerre polynomials, $L_n^{(m')}(z)$. The radial wave function can be expressed as

$$g(r) = A_{n,m}r^{|m'|}\exp(-\gamma r^2/2)L_n^{(m')}(\gamma r^2), \tag{52}$$

where $A_{n,m}$ is the normalization constant. The relation for the orthogonality of Laguerre polynomials is (Abramowitz and Stegun, 1964)

$$\int_0^\infty z^c e^{-z}L_n^{(c)}(z)L_{n'}^{(c)}(z)dz=\frac{\Gamma(n+c+1)}{n!}\delta_{n,n}. \tag{53}$$

from which one can obtain

$$A_{n,m}=\sqrt{\frac{2\gamma^{m'+1}n!}{\Gamma(n+m'+1)}}. \tag{54}$$

which is the normalization constant.

References

Abramowitz, M., Stegun, I.A., 1964. Handbook of Mathematical Functions. Dover, New York.
Akcay, H., Tezcan, C., 2009. Exact solutions of the Dirac equation with harmonic oscillator potential including Coulomb-like tensor potential. Int. J. Mod. Phys. C 20 (6), 931–940.
Alhaidari, A.D., Bahlouli, H., Al-Hasan, A., 2006. Dirac and Klein–Gordon equations with equal scalar and vector potentials. Phys. Lett. A 349, 87–97.
Aygun, M., Bayrak, O., Boztosun, I., Sahin, Y., 2012. The energy eigenvalues of the Kratzer potential in the presence of a magnetic field. Eur. Phys. J.D. 66, 35.
Bogachek, E.N., Landman, U., 1995. Edge states, Aharonov–Bohm oscillations and thermodynamic and special properties in a two-dimensional electron gas with an antidot. Phys. Rev. B 52, 14067–14077.
Çetin, A., 2008. A quantum pseudodot system with a two-dimensional pseudoharmonic potential. Phys. Lett. A 372, 3852–3856.
Child, M.S., Dong, S.-H., Wang, X.-G., 2000. Quantum states of a sextic potential hidden symmetry and quantum monodromy. J. Phys. A Math. Gen. 33, 5653.
Dong, S.-H., Ma, Z.-Q., 1998. Exact solutions of the Schrödinger equation for the potential $V(r) = ar^2 + br^{-2} + cr^{-6}$ in two-dimensions. J. Phys. A Math. Gen. 31, 9855.
Dong, S.-H., 2000. Exact solutions of the two-dimensional Schrödinger equation with certain central potentials. Int. J. Theor. Phys. 39, 1119–11128.
Dong, S.-H., 2001a. Schrödinger equation with the potential $V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1}$. Phys. Scr. 64, 273.
Dong, S.-H., 2001b. A new approach to the relativistic Schrödinger equation with central potentials: Ansatz method. Int. J. Theor. Phys. 40, 567–569.
Dong, S.-H., 2002. On the solutions of the Schrödinger equation with some Anharmonic potentials: wave function ansatz. Phys. Scr. 65, 289–295.
Falaye, B.J., 2012a. The Klein–Gordon equation with ring-shaped potentials: asymptotic iteration method. J. Math. Phys. 53, 082107.
Falaye, B.J., 2012b. Exact solutions of the Klein–Gordon equation for spherically asymmetrical singular oscillator. Few-Body Syst. 53, 563571.
Falaye, B.J., 2012c. Any ℓ-state solutions of the Eckart potential via asymptotic iteration method. Cent. Eur. J. Phys. 10, 960–965.
Ferkous, N., Bounames, A., 2004. Energy spectrum of a 2D Dirac oscillator in the presence of the Aharonov–Bohm effect. Phys. Lett. A 325, 21–29.
Flügge, S., 1994. Practical Quantum Mechanics 1. Springer, Berlin.
Ginocchio, J.N., 2005. U(3) and pseudo-U(3) symmetry of the relativistic harmonic oscillator. Phys. Rev. Lett. 95, 252501.
Greiner, W., Müller, B., 1994. Quantum Mechanics: An Introduction. Springer, Berlin.
Greiner, W., 2000. Relativistic Quantum Mechanics: Wave Equations. Springer-Verlag, Berlin.
Ikhdair, S.M., Sever, R., 2007. Exact polynomial eigensolutions of the Schrödinger equation for the pseudoharmonic potential. J. Mol. Struct. (Theochem.) 806, 155–158.
Ikhdair, S.M., Sever, R., 2008a. Polynomial solutions of the Mie-type potential in D-dimensional Schrödinger equation. J. Mol. Struct. (Theochem.) 855, 13–17.
Ikhdair, S.M., Sever, R., 2008b. Exact solutions of the D-dimensional Schrö dinger equation for a ring-shaped pseudoharmonic potential. Cent. Eur. J. Phys. 6 (3), 685–696.
Ikhdair, S.M., Sever, R., 2008c. Exact bound states of the D-dimensional Klein–Gordon equation with equal scalar and vector ring-shaped pseudoharmonic potential. Int. J. Mod. Phys. C 19 (9), 1425–1442.
Ikhdair, S.M., Sever, R., 2009. Exactly solvable effective mass D-dimensional Schrödinger equation for pseudoharmonic and modified Kratzer potentials. Int. J. Mod. Phys. C 20 (3), 361–379.
Ikhdair, S.M., 2009. Rotational and vibrational diatomic molecule in the Klein–Gordon equation with hyperbolic scalar and vector potentials. Int. J. Mod. Phys. C 20, 1563–1582.
Ikhdair, S.M., 2012. Exact solutions of Dirac equation with charged harmonic oscillator in electric field: bound states. J. Mod. Phys. 3, 170–179.
Ikhdair, S.M., Hamzavi, M., Sever, R., 2012. Spectra of cylindrical quantum dots: the effect of electrical and magnetic fields together with Aharonov–Bohm flux fields. Phys. B Condens. Mater. Phys. 407, 4523–4529.
Ikhdair, S.M., Hamzavi, M., 2012a. A quantum pseudodot system with two-dimensional pseudoharmonic oscillator in external magnetic and Aharonov–Bohm fields. Phys. B Condens. Mater. Phys. 407, 4198–4207.
Ikhdair, S.M., Hamzavi, M., 2012b. Effects of external fields on a two-dimensional Klein–Gordon particle under pseudoharmonic oscillator interaction. Chin. Phys. B 21, 110302–110306.

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Ikhdair, S.M., Hamzavi, M., 2012c. Spectral properties of quantum dots influenced by a confining potential model. Phys. B Condens. Mater. Phys. 407, 4797–4803.

Ikhdair, S.M., 2013. Scalar charged particle in presence of magnetic and Aharonov–Bohm fields plus scalar–vector Killingbeck potentials. Few-Body Syst. http://dx.doi.org/10.1007/s00601-013-0693-2.

Khordad, R., 2010. Effects of magnetic field and geometrical size on the interband light absorption in a quantum pseudodot system. Solid State Sci. 12, 1253–1256.

Khordad, R., 2011. Temperature effect on the threshold frequency of absorption in a quantum pseudodot. Phys. B Condens. Mater. Phys. 406, 620–623.

Liboff, R.L., 2003. Introductory Quantum Mechanics. Pearson Education–Addison Wesley, San Francisco, CA, p. 431, 528.

Lisboa, R., Malheiro, M., de Castro, A.S., Alberto, P., Fiolhais, M., 2004. Pseudospin symmetry and the relativistic harmonic oscillator. Phys. Rev. C 69, 024319.

Lu, F.L., Chen, C.Y., Sun, D.S., 2005. Bound states of Klein–Gordon equation for double ring-shaped oscillator scalar and vector potentials. Chin. Phys. 14, 463–467.

Nikiforov, A.F., Uvarov, V.B., 1988. Special Functions of Mathematical Physics. Birkhauser, Berlin.

Qiang, W.-C., 2004. Bound states of the Klein–Gordon equation for ring-shaped Kratzer-type potential. Chin. Phys. 13, 575–578.

Rajabi, A.A., Hamzavi, M., 2013. Relativistic effect of external magnetic and Aharonov–Bohm fields on the unequal scalar and vector Cornell model. Eur. Phys. J. Plus 128, 5–6.

Tezcan, C., Sever, R., 2009. A general approach for the exact solution of the Schrödinger equation. Int. J. Theor. Phys. 48, 337–350.

Wang, I.C., Wong, C.Y., 1988. Finite-size effect of the Schrödinger particle–production mechanism. Phys. Rev. D 38, 348–359.

Wybourne, B.G., 1974. Classical Groups for Physicists. Wiley, Canada.

Xu, Y., He, S., Jia, C.-S., 2010. Approximate analytical solutions of the Klein–Gordon equation with the Pöschl–Teller potential including the centrifugal term. Phys. Scr. 81, 045001.

Zhang, F.L., Song, C., Chen, J.-L., 2009. Dynamical symmetries of two-dimensional systems in relativistic quantum mechanics. Ann. Phys. 324, 173–177.

Zhang, F.-L., Chen, J.-L., 2009. Dynamical symmetries of the Klein–Gordon equation. J. Math. Phys. 50, 032301–032307.

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