



Eigenspectra and statistical properties of the Klein–Gordon equation with Cornell potential: Unequal mixings of scalar and time-like vector potentials

F. Tajik^a, Z. Sharifi^a, M. Eshghi^{b,*}, M. Hamzavi^c, M. Bigdeli^a, S.M. Ikhdair^{d,e}

^a Department of Physics, University of Zanjan, Zanjan, Iran

^b Department of Physics, Imam Hossein Comprehensive University, Tehran, Iran

^c Department of Mathematics and Statistics, University of Texas at El Paso, El Paso, TX, USA

^d Department of Physics, Faculty of Science, An-Najah National University, Nablus, West Bank, Palestine

^e Department of Electrical Engineering, Near East University, Nicosia, Northern Cyprus, Mersin 10, Turkey

ARTICLE INFO

Article history:

Received 7 April 2019

Received in revised form 26 April 2019

Available online 20 August 2019

Keywords:

Klein–Gordon equation

Cornell potential

Laplace transform method

Statistical properties

ABSTRACT

The D-dimensional Klein–Gordon (KG) wave equation with unequal scalar and time-like vector Cornell interactions was solved by the Laplace transform method. We have obtained the bound state energy eigenvalues of the spinless relativistic heavy quarkonium systems under such a potential. Further, the stationary states are calculated due to the well behavior of wave functions at the origin and at infinity. The statistical properties of this model were also investigated. Our results have been found to be of great importance in particle physics.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

Relativistic wave equations such as the Dirac and the Klein–Gordon (KG) equations have been received much interest from many theoretical physicists in different fields of physics [1,2]. The particle dynamics in relativistic quantum mechanics is explained by using KG and Dirac equations.

On the other hand, the most interesting choices for the investigated potentials are probably the spherically symmetric ones because of their wide applications in several areas of physics including nuclear and particle physics. For example, solutions of the wave equations are extremely significant particularly in describing the nuclear shell structure [3–8]. So far a variety of analytical methods are expanded and utilized in solving these wave equations. As an example, see; the Supersymmetry (SUSY) technique [9–11], the Nikiforov–Uvarov method (NU) [12–14], the asymptotic iteration method (AIM) [15–19], the variational [20], the hypervirial perturbation method [21], the shifted large $1/N$ expansion (SE), the modified shifted $1/N$ expansion (MSE) [22], the exact quantization rule (EQR) [23–25], the Laplace transform (LT) [26–28], the perturbative formalism [29,30], the polynomial solution [31], the wave function ansatz method [32], the group theory [33,34], the path integral approach [35–38], L^2 approach [39], the proper quantization rule [40–43] and the point canonical transformation (PCT) [44–47].

The LT method is extensively used to solve the quantum wave equations. It was absolutely used during the initial years of quantum mechanics by Schrödinger when discussing the radial eigenfunctions of the hydrogen atom [48] and later Enfield approached the Schrödinger equation with the Coulomb, harmonic oscillator, exponential and Yamauchi

* Corresponding author.

E-mail address: eshgi54@gmail.com (M. Eshghi).

potentials [49]. The hydrogen atom was also investigated using the LT method [50]. In the last years, the $1/x$ [51], Morse [52], D-dimensional harmonic oscillator [53], 3D pseudo-harmonic and Mie-type [54], one-dimensional harmonic oscillator [55], the Mie-type [26,27], the pseudo-harmonic oscillatory ring-shaped [28], and the Makarov [56] potentials were solved by means of LT method.

Nowadays, for spin zero particles, the KG equation is regarded as the relativistic form of the Schrödinger equation. In the other word, among various nonrelativistic and relativistic wave equations, the KG equation has been receiving much theoretical and phenomenological attention as it allows us to investigate spin-zero particles. The KG equation has been used to mathematical physics such as nonlinear optics, solid-state physics, and quantum field theory. As an example, Grave et al. have considered three examples of vibrating systems whose mathematical explanations lead to the KG equation [57]. Also, recently, many authors have been investigated the KG equation with various kinds of potentials by using various methods to describe the corresponding relativistic physical systems [58–71]. Within the framework of the Klein–Gordon equation with some potential energy models in higher spatial dimensions, some authors have investigated the spinless energies for some real diatomic molecules [72–74].

Here, we investigate the bound state energy eigenvalues and the wave functions of the D-dimensional KG equation with unequal mixings of the vector–scalar Cornell potentials by means of the LT approach. Further, the energy states are mainly used to obtain the partition function that is essentially used to study the essential statistical properties like the thermal quantities of the Cornell model.

This paper is organized as follows. In Section 2, we study the bound state energy eigenvalues of the spinless relativistic heavy quarkonium systems under unequal mixings of scalar and vector Cornell potentials using the LT technique. In brief, we also give in brief a review to the LT method in this section. We obtain the exact bound state energy eigenvalues and the corresponding wave functions of the D-dimensional KG equation with Cornell potential. In Section 3, we study the statistical properties and the thermal quantities such as the Helmholtz free energy, the mean energy, the entropy, and the specific heat. Finally, we give some of our conclusions in Section 4.

2. Bound state energies

In spherical coordinates, the KG equation with time-like vector $V(r)$ and scalar $S(r)$ potentials can be written as follows (in units of $\hbar = c = 1$)

$$[\Delta_D + (E_{nl} - V(r))^2 - (M + S(r))^2] \Psi_{nlm}(r, \Omega_D) = 0, \quad (1)$$

with

$$\Delta_D = \nabla_D^2 = r^{1-D} \frac{\partial}{\partial r} (r^{D-1} \frac{\partial}{\partial r}) - \frac{\Lambda_D^2(\Omega_D)}{r^2}, \quad (2)$$

where $\Lambda_D^2(\Omega_D)$ is the hyperangular momentum operator [37,47] reads as below

$$\Lambda_D^2 = - \sum_{i,j=1}^D \Lambda_{ij}^2, \quad \Lambda_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad (3)$$

including all Cartesian components x_i of the D-dimensional vector (x_1, x_2, \dots, x_D) . The eigenvalues $\Lambda_D^2(\Omega_D)$ are written by [9]

$$\Lambda_D^2(\Omega_D) Y_l^m(\Omega_D) = \frac{l(l+D-2)}{r^2} Y_l^m(\Omega_D), \quad D > 1 \quad (4)$$

where $Y_l^m(\Omega_D)$ is the hyperspherical harmonics. For the case $D = 3$, we have a familiar form of (4) as $\Lambda_{D=3}^2(\Omega_{D=3}) Y_l^m(\Omega_{D=3}) = \frac{l(l+1)}{r^2} Y_l^m(\theta, \phi)$. Now, after employing the separation of variables using the following wave function $\Psi_{nlm}(r, \Omega_D) = R_{nl}(r) Y_l^m(\Omega_D)$, Eq. (1) simplifies as below:

$$\left[\frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} - (M^2 - E_{nl}^2) + V^2(r) - S^2(r) - 2(E_{nl}V(r) + MS(r)) - \frac{l(l+D-2)}{r^2} \right] R_{nl}(r) = 0, \quad (5)$$

with unequal mixture of scalar and time-like vector Cornell potentials taking the simple form:

$$V(r) = -\frac{a_v}{r} + b_v r, \quad (6a)$$

$$S(r) = -\frac{a_s}{r} + b_s r. \quad (6b)$$

Upon substituting the above potentials, then Eq. (5) turn to as follows:

$$\left[\frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} + \frac{a_v^2 - a_s^2 - l(l+D-2)}{r^2} + \frac{2(Ea_v + Ma_s)}{r} + (b_v^2 - b_s^2)r^2 - 2(Eb_v + Mb_s)r - \varepsilon^2 - 2(a_v b_v - a_s b_s) \right] R_{nl}(r) = 0, \tag{7}$$

where $\varepsilon^2 = M^2 - E_{nl}^2$. In the limit when $r \rightarrow \infty$, the wave function satisfies the following differential equation:

$$\left(\frac{d^2}{dr^2} - \beta^2 r^2 \right) R_{nl}(r) = 0, \tag{8}$$

where we have considered the fact $(b_v^2 - b_s^2)r^2 \gg -2(Eb_v + Mb_s)r$, for larger r and assigned

$$\beta = \sqrt{b_s^2 - b_v^2}, b_s^2 > b_v^2. \tag{9}$$

The solution of the Eq. (8) in the limiting case when $r^2 \gg 1$ is easy, with the solution as $R_{nl}(r) \propto e^{-\frac{\beta r^2}{2}}$.

Now, we seek a complete general solution to Eq. (7) which needs to be bounded, well defined at the origin and finite at the boundaries. Let us look for a solution in its general form as

$$R_{nl}(r) = r^k e^{-\frac{\beta r^2}{2}} f(r), k > 0, \tag{10}$$

The term r^k assures us that the solution at $r = 0$ is bounded. Now, we try to find the function $f(r)$. The first and second order derivatives of Eq. (10) will be as below

$$\frac{dR_{nl}}{dr} = r^k e^{-\frac{\beta r^2}{2}} \left\{ \frac{df(r)}{dr} + \frac{k}{r} f(r) - \beta r f(r) \right\}, \tag{11}$$

and

$$\frac{d^2 R_{nl}}{dr^2} = r^k e^{-\frac{\beta r^2}{2}} \left\{ \frac{d^2 f(r)}{dr^2} + \left(\frac{2k}{r} - 2\beta r \right) \frac{df(r)}{dr} + \left(\frac{k(k-1)}{r^2} - 2\beta k - \beta + \beta^2 r^2 \right) f(r) \right\}, \tag{12}$$

respectively.

Inserting them back into Eq. (7), we can obtain the following equation as

$$\left\{ r f'(r) + \frac{\{ (2k+D-1) - 2\beta r^2 \} f'(r) + \{ k(k-1) + k(D-1) - l(l+D-2) + (a_v^2 - a_s^2) \}}{r} - 2(Eb_v + Mb_s)r^2 + 2(Ea_v + Ma_s) + (-2\beta k - \beta D - \varepsilon^2 - 2(a_v b_v - a_s b_s))r \right\} f(r) = 0, \tag{13}$$

where the prime over $f(r)$ denotes the derivative with respect to r . To apply the LT method on the above differential equation we must impose the following parametric condition

$$k(k-1) + k(D-1) - l(l+D-2) + (a_v^2 - a_s^2) = 0. \tag{14}$$

Under the above condition, Eq. (13) reduces into a rather more simplified form as

$$r f'(r) + \{ (2k+D-1) - 2\beta r^2 \} f'(r) + \{ -2(Eb_v + Mb_s)r^2 + 2(Ea_v + Ma_s) + (-2\beta k - \beta D - \varepsilon^2 - 2(a_v b_v - a_s b_s))r \} f(r) = 0. \tag{15}$$

Let us now implement the main basic ideas of the LT method so that we can solve the above differential equation [53–55,75–77]. It is simply an integral transform and comprehensively useful in both physics and engineering. Recently, LT method has been utilized by many authors to solve the Schrödinger equation for various potential models [48,50,52,55,75–77]. The advantage of this method is that it converts the second-order differential equation into a first-order differential equation whose solutions may be obtained simply. We trace Refs. [78,79] to display the basic ideas of the LT method as follows:

$$\phi(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt, \tag{16a}$$

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0), \tag{16b}$$

$$L\{t^n f(t)\} = (-1)^n \phi^{(n)}(s), \tag{16c}$$

$$L\{t f(t)\} = -\frac{d\phi(s)}{ds}, \tag{16d}$$

$$L\{tf'(t)\} = -2s\phi(s) - s^2 \frac{d\phi(s)}{ds} + f(0); f(0) = 0, \quad (16e)$$

$$L\{tf'(t)\} = s\phi(s), \quad (16f)$$

where the superscript (n) denotes the n th derivative with respect to t for $f^{(n)}(t)$, and with respect to s for $\phi^{(n)}(s)$. Furthermore, taking the following boundary condition $f(0) = 0$, Eq. (15) reduces to the following equation

$$(s + \alpha) \frac{d^2\phi(s)}{ds^2} + \left(\frac{1}{2\beta} s^2 + \lambda \right) \frac{d\phi(s)}{ds} + \left(\gamma s - \frac{(Ea_v + Ma_s)}{\beta} \right) \phi(s) = 0, \quad (17)$$

where the following identifications are used as below

$$\alpha = \frac{(Ea_v + Ma_s)}{\beta}; \lambda = -\frac{\varepsilon^2}{2\beta} + \left(-k - \frac{D}{2} + 2 \right) - \frac{(a_v b_v - a_s b_s)}{\beta}; \gamma = \frac{3 - 2k - D}{2\beta}, \quad (18)$$

with $s = -\alpha$ is the singular point of the Eq. (17).

If there is some constant $\sigma \in R$ such that $|e^{-\sigma t} f(t)| \leq M$ for sufficiently large t , then the integral in Eq. (16a) will exist for $\text{Re } s > \sigma$. The LT may fail to exist because of a sufficiently strong singularity in the function $f(t)$ as $t \rightarrow 0$. In particular, we have

$$L \left[\frac{t^\alpha}{\Gamma(\alpha + 1)} \right] = \frac{1}{s^{\alpha+1}}, \alpha > -1. \quad (19)$$

Notice that if s_0 be as the singular point, then the LT behaves as $s \rightarrow s_0$

$$\phi(s) = \frac{1}{(s - s_0)^\nu}. \quad (20)$$

Then for the limiting case when $t \rightarrow \infty$, we have

$$f(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} e^{s_0 t}, \quad (21)$$

where $\Gamma(\nu)$ is the gamma function. So, applying the fact of Eq. (20) the solution of Eq. (17) may be taken

$$\phi(s) = \frac{C_{nID}}{(s + \alpha)^{n+1}}, s_0 = \alpha, \nu = n + 1, \quad n = 0, 1, 2, 3, \dots \quad (22)$$

Plugging Eq. (22) in Eq. (17) we can obtain the following three identity relations,

$$\gamma = \frac{n + 1}{2\beta}, \quad (23)$$

$$\alpha = \frac{Ea_v + Ma_s}{\beta}, \beta = \sqrt{b_s^2 - b_v^2}, b_s \geq b_v \quad (24)$$

$$(n + 1)(n + 2) - (n + 1)\lambda - \frac{(Ea_v + Ma_s)\alpha}{\beta} = 0. \quad (25)$$

Making use of Eqs. (23)–(25) together with Eq. (18), the energy eigenvalues will be as

$$E^2 - \frac{2Mb_v}{b_s} E - \left(1 - \frac{b_v^2}{b_s^2} \right) \left[M^2 + 2 \left(a_v b_v - a_s b_s + \left(k + n + \frac{D}{2} \right) \sqrt{b_s^2 - b_v^2} \right) \right] - M^2 = 0. \quad (26)$$

where k can be found from the Eq. (14) as below

$$k = (2 - D) \pm \sqrt{(D - 2)^2 - 4[(a_v^2 - a_s^2) - l(l + D - 2)]}. \quad (27)$$

We can calculate the numerical results of the energy levels for positive sign, (+) for the various dimensional at the different values in Table 1.

On the other hand, the undetermined wave function $f(r)$ can be obtained from the inverse LT such that $f(r) = L^{-1}\{\phi(s)\}$ [45,46]. Hence, using the Eq. (21) with $\nu = n + 1$ and $s_0 = -\alpha$, we get

$$f(r) = \frac{C_{nID}}{n!} r^n e^{-\alpha r}. \quad (28)$$

Finally, with the help of the Eq. (28), the wave function of this system in Eq. (10) can be simply determined as

$$R_{nID}(r) = \frac{C_{nID}}{n!} r^{k+n} \exp \left(-\sqrt{b_s^2 - b_v^2} \frac{r^2}{2} - \left(\frac{Ea_v + Ma_s}{\sqrt{b_s^2 - b_v^2}} \right) r \right). \quad (29)$$

Table 1

Relativistic energy of the system for different states and dimensions with $M = 1, b_v = 0.002, b_s = 2, a_v = 0.2, a_s = 6$.

	l	$E_{n,l}^D (D = 1)$	$E_{n,l}^D (D = 2)$	$E_{n,l}^D (D = 3)$	$E_{n,l}^D (D = 4)$	$E_{n,l}^D (D = 5)$	$E_{n,l}^D (D = 6)$	
$n = 1$	0	6.012	5.829	5.670	5.536	5.427	5.346	
		-6.010	-5.827	-5.668	-5.534	-5.425	-5.344	
$n = 2$	0	6.336	6.163	6.012	5.886	5.784	5.708	
		-6.334	-6.161	-6.010	-5.884	-5.782	-5.706	
	1	6.336	6.108	5.899	5.712	5.550	5.414	
		-6.334	-6.106	-5.897	-5.710	-5.548	-5.412	
$n = 3$	0	6.644	6.479	6.336	6.216	6.120	6.048	
		-6.642	-6.477	-6.334	-6.214	-6.118	-6.046	
	1	6.644	6.427	6.229	6.052	5.899	5.771	
		-6.642	-6.425	-6.227	-6.050	-5.897	-5.769	
	2	6.542	6.264	5.998	5.750	5.522	5.321	
		-6.540	-6.262	-5.996	-5.748	-5.520	-5.319	
$n = 4$	0	6.939	6.781	6.644	6.530	6.439	6.370	
		-6.937	-6.779	-6.642	-6.528	-6.437	-6.368	
	1	6.939	6.731	6.542	6.374	6.229	6.108	
		-6.937	-6.729	-6.540	-6.372	-6.227	-6.106	
	2	6.841	6.575	6.323	6.088	5.873	5.684	
		-6.839	-6.573	-6.321	-6.086	-5.871	-5.682	
	3	6.632	6.289	5.947	5.612	5.289	4.984	
		-6.630	-6.287	-5.945	-5.610	-5.287	-4.982	
	$n = 5$	0	7.221	7.070	6.939	6.830	6.742	6.677
			-7.219	-7.068	-6.937	-6.828	-6.740	-6.675
		1	7.221	7.022	6.841	6.680	6.542	6.427
			-7.219	-7.020	-6.839	-6.678	-6.540	-6.425
2		7.127	6.873	6.632	6.408	6.205	6.026	
		-7.125	-6.871	-6.630	-6.406	-6.203	-6.024	
3		6.927	6.599	6.275	5.958	5.654	5.370	
		-6.925	-6.597	-6.273	-5.956	-5.652	-5.368	
4		6.586	6.144	5.676	5.177	4.644	4.071	
		-6.584	-6.142	-5.674	-5.175	-4.642	-4.069	

where k is given by Eq. (27). To obtain the normalization constant C_{nID} we use the condition [80,81] as below:

$$\int_0^\infty [R_{nID}(r)]^2 r^{D-1} dr = 1.$$

A fair approximation $r + \frac{\alpha}{2\beta} \approx r$ would make the integration much easier to evaluate. As a consequence, the use of the following integral formula

$$\int_0^\infty x^p e^{-Ax^q} dx = \frac{1}{q} \frac{\Gamma(h)}{A^h}, p, q > 0, \tag{30a}$$

where $h = \frac{p+q}{q}$ leads to

$$C_{nID} = n! \sqrt{\frac{2(\beta)^{k+n+D/2}}{\Gamma(k+n+D/2)}}. \tag{30b}$$

3. Statistical properties of the Cornell potential

We can see that a few authors have obtained the results for predicting the thermochemical quantities of the systems from the non-relativistic region to relativistic region for actual substances. For example, see [82–86].

To investigate the statistical properties, we need to obtain the partition function of the system using [87]

$$Z(\beta) = \sum_{i=0}^n e^{-\beta(E_n - E_0)}, \tag{31}$$

where $\beta = 1/k_B T$, k_B is the Boltzmann constant and T is the temperature in Kelvin.

We start by recasting Eq. (26) as follows

$$E_n = M \frac{b_v}{b_s} \pm M [\alpha + 4n\delta]^{\frac{1}{2}}, \tag{32}$$

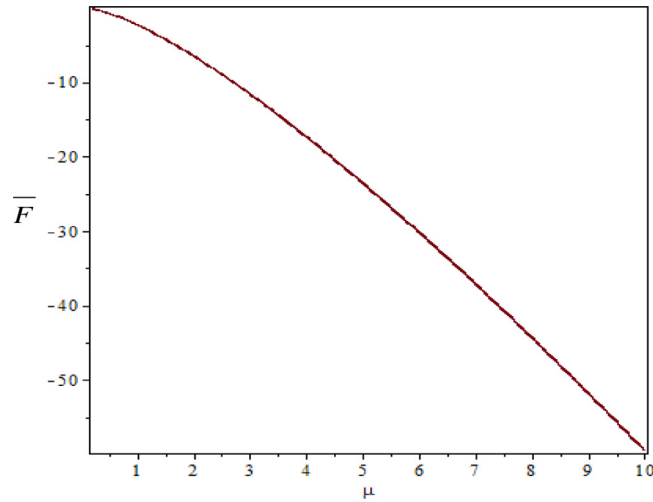


Fig. 1. The free energy (\bar{F}) for the Cornell potential versus μ .

with the identifications $\alpha = \frac{1}{2M^2} \sqrt{b_s^2 - b_v^2} \left(1 - \frac{b_v^2}{b_s^2}\right)$ and $\delta = \frac{2}{M^2} \left(1 - \frac{b_v^2}{b_s^2}\right) \left(a_v b_v - a_s b_s + \sqrt{b_s^2 - b_v^2} \left(k + \frac{D}{2}\right)\right)$, then using the Euler–McLaurin formula [87,88],

$$\sum_{m=0}^{\infty} f(m) = \frac{1}{2} f(0) + \int_0^{\infty} f(x) dx - \sum_{i=1}^{\infty} \frac{1}{(2i)!} B_{2i} f^{(2i-1)}(0), \quad (33)$$

where B_{2i} are the Bernoulli numbers, $B_2 = 1/6$, $B_4 = -1/30$, ..., taking up to $i = 2$, the canonical partition function be obtained as follows

$$Z(\mu) = \frac{1}{2} + \frac{\mu}{2\alpha} \left(\mu + \sqrt{\delta}\right) + \frac{\alpha^3}{60\mu^3 \sqrt{\delta^3}} \left(\left(\frac{\delta}{\alpha^3} - \frac{2}{\delta}\right) \mu^2 - \frac{2}{\sqrt{\delta}} \mu - 1\right), \quad (34)$$

where $\mu = \frac{1}{M\beta} = \frac{k_B T}{M}$. In what follows, we need to study thermal quantities such as the Helmholtz free energy, the mean energy, the entropy and the specific heat as below

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta), \quad U(\beta) = -\frac{\partial}{\partial \beta} \ln Z(\beta), \quad (35a)$$

$$S(\beta) = k_B \beta^2 \frac{\partial}{\partial \beta} F(\beta), \quad C_v(\beta) = -k_B \beta^2 \frac{\partial}{\partial \beta} U(\beta), \quad (35b)$$

It would be better to rewrite the above quantities as a function of $\mu = \frac{1}{M\beta}$ in the following forms:

$$\bar{F}(\mu) = \frac{F}{M} = -\mu \ln Z(\mu), \quad \bar{U}(\mu) = \frac{U}{M} = \mu^2 \frac{\partial}{\partial \mu} \ln Z(\mu), \quad (36a)$$

$$\bar{S}(\mu) = \frac{S}{K_B} = \ln Z(\mu) + \mu \frac{\partial}{\partial \mu} \ln Z(\mu), \quad (36b)$$

$$\bar{C}_v(\mu) = \frac{C_v}{K_B} = 2\mu \frac{\partial}{\partial \mu} \ln Z(\mu) + \mu^2 \frac{\partial^2}{\partial \mu^2} \ln Z(\mu), \quad (36c)$$

It is noteworthy that for the limiting case at high temperatures corresponding to $\beta \ll 1$, our results reduce to

$$Z(\mu) \simeq \mu^2, \quad U(\mu) \simeq 2\mu, \quad C \simeq 2, \quad (37)$$

These results can be easily inspected from the Figs. 1–4.

4. Results

The vibrational and rotational bound state energies of scalar and time-like vector Cornell potential have been calculated for various n and l quantum numbers in D -dimensions.

The obtained energy values are showed in Table 1 for $n = 1, 2, 3, 4, 5$, $l = 0, 1, 2, 3, 4$ and $D = 1, 2, \dots, 6$ in order to give a wider energy spectrum. We see that the energy levels are decreasing with increasing dimensional and angular momentum, but it is increasing with increasing quantum numbers.

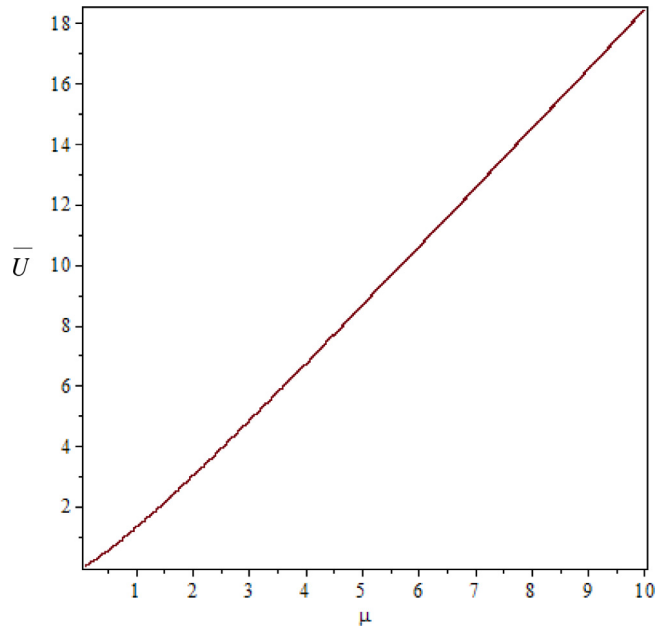


Fig. 2. The mean energy (\bar{U}) for the Cornell potential versus μ .

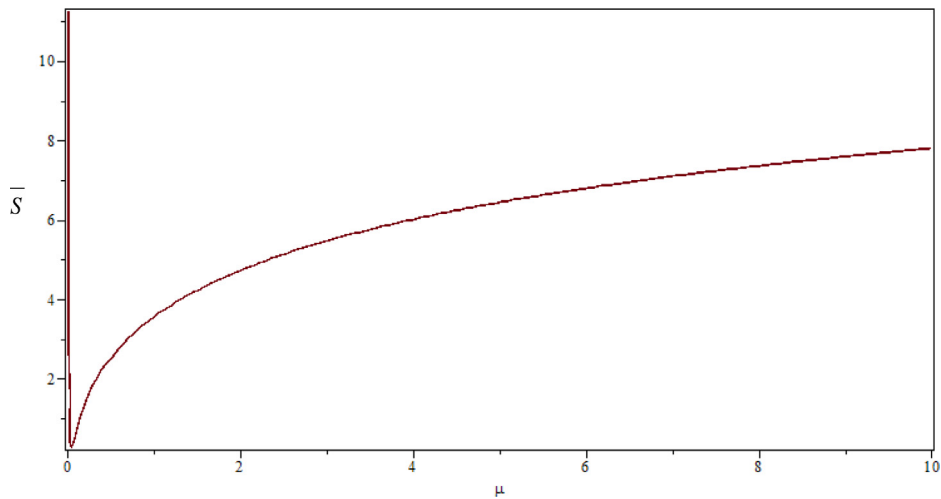


Fig. 3. The entropy (\bar{S}) for the Cornell potential versus μ .

We also presented the behavior of the thermodynamics properties as a function of the potential parameters. In fact, our results have been showed in Figs. 1–4 where we have plotted the variation of all thermal quantities versus the dimensionless parameter μ for the Cornell potential. In Fig. 1, we see that the Helmholtz free energy decreases exponentially with increasing of μ , but, in Fig. 2, the mean energy increases linearly. Also, the entropy decrease with increasing the parameter μ in Fig. 3. In fact, at the beginning, it decreases with the increasing μ , then it reaches a minimum at around $\mu = 0$, after the entropy begins to increase exponentially. In Fig. 4, the specific heat capacity increases exponentially with increasing the dimensionless parameter μ .

5. Conclusions

We have investigated the solution of the D-dimensional KG wave equation with unequal couplings of scalar and time-like vector Cornell potentials via the LT method. As the Cornell potential plays an important role in studying of the quarkonium and baryonic systems, so we have solved the KG equation with it and obtained the exact bound state energy eigenvalues and the corresponding wavefunctions. The statistical properties of the present model have been

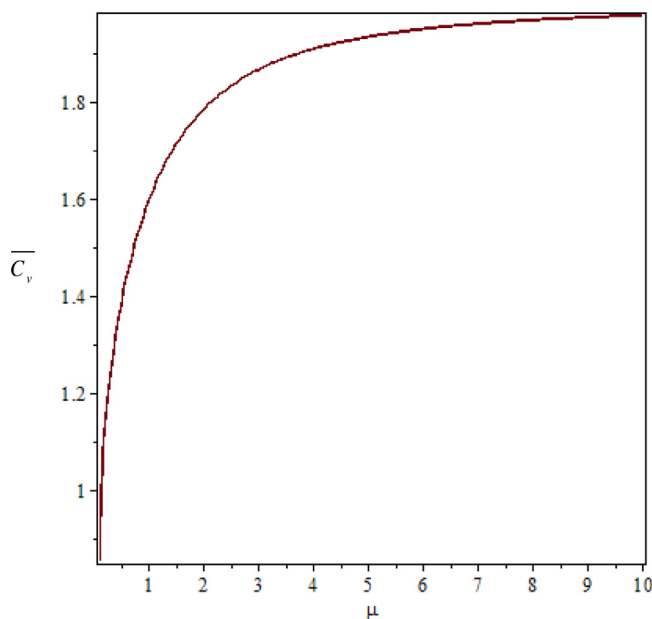


Fig. 4. The specific heat (\overline{C}_v) for the Cornell potential versus μ .

investigated and the thermal quantities such as the Helmholtz free energy, the mean energy, the entropy and the specific heat have been plotted as shown in Figs. 1–4. The behaviors of the thermal quantities versus the dimensionless parameter $\mu = 1/M\beta$ were obtained at high temperature. Finally, our result reduces into the limit as a function of $\overline{\mu}$ as follows $Z(\mu) \simeq \mu^2$, $U(\mu) \simeq 2\mu$, $C \simeq 2$.

References

- [1] A.W. Thomas, W. Weise, *Structure of the Nucleon*, Wiley-VCH, Berlin, 2001.
- [2] B. Thaller, *The Dirac Equation*, Springer-Verlag, New York, 1992.
- [3] C.S. Jia, P. Guo, X.L. Peng, *J. Phys. A: Math.* 39 (2006) 7737.
- [4] R. Sever, C. Tezcan, M. Aktas, O. Yesiltas, *J. Math. Chem.* 43 (2007) 845.
- [5] S.M. Ikhdaïr, M. Hamzavi, *Chin. Phys. B* 21 (2012) 110302.
- [6] M.R. Pahlavani, S.M. Motevalli, *Int. J. Theor. Phys.* 48 (2009) 1622.
- [7] S.M. Ikhdaïr, R. Sever, *Internat. J. Modern Phys. C* 20 (2008) 361.
- [8] S.M. Ikhdaïr, R. Sever, *Internat. J. Modern Phys. E* 17 (2008) 1107.
- [9] F. Cooper, A. Khare, U. Sukhatme, *Phys. Rep.* 251 (1995) 267.
- [10] D.A. Morales, *Chem. Phys. Lett.* 394 (2004) 68.
- [11] S.H. Dong, *Factorization Method in Quantum Mechanics*, Springer, Netherlands, 2007.
- [12] M. Eshghi, S.M. Ikhdaïr, *Math. Methods Appl. Sci.* 37 (2014) 2829.
- [13] M. Eshghi, H. Mehraban, S.M. Ikhdaïr, *Pramana J. Phys.* 88 (2017) 73.
- [14] A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Basel, 1988.
- [15] M. Eshghi, H. Mehraban, *Chin. J. Phys.* 54 (4) (2012) 533.
- [16] M. Eshghi, *Acta Sci. Tech.* 34 (2) (2012) 207.
- [17] O. Bayrak, I. Boztosun, *J. Phys. A: Math. Gen.* 39 (2006) 6955.
- [18] H. Ciftci, R.L. Hall, N. Saad, *Phys. Lett. A* 340 (2005) 388.
- [19] M. Eshghi, R. Sever, S.M. Ikhdaïr, *Eur. Phys. J. Plus* 131 (2016) 223.
- [20] E.D. Filho, R.M. Ricotta, *Phys. Lett. A* 269 (2000) 269.
- [21] J.B. Killingbeck, A. Grosjean, G. Jolicard, *J. Chem. Phys.* 116 (2002) 447.
- [22] M. Bag, M.M. Panja, R. Dutt, *Phys. Rev. A* 46 (1992) 6059.
- [23] Z.Q. Ma, B.W. Xu, *Europhys. Lett.* 69 (2005) 685.
- [24] W.C. Qiang, S.H. Dong, *Eur. Phys. Lett.* 89 (2010) 10003.
- [25] W.C. Qiang, G.H. Sun, S.H. Dong, *Ann. Phys.* 524 (2012) 360.
- [26] M. Eshghi, S.M. Ikhdaïr, *Z. Naturforsch.* 69a, 2014, p. 111.
- [27] M. Eshghi, S.M. Ikhdaïr, *Chin. Phys. B* 23 (12) (2014) 120304.
- [28] M. Eshghi, *Chin. Phys. Lett.* 29 (11) (2012) 110304.
- [29] B. Gönül, K. Köksal, E. Bakir, *Phys. Scr.* 73 (2006) 279.
- [30] S.M. Ikhdaïr, R. Sever, *Internat. J. Modern Phys. A* 21 (2006) 6465.
- [31] S.M. Ikhdaïr, R. Sever, *J. Mol. Struct.* 806 (2007) 155.
- [32] S.H. Dong, *Internat. J. Theoret. Phys.* 40 (2001) 569.
- [33] S.H. Dong, *Phys. Scr.* 65 (2002) 289.
- [34] Y. Alhassid, F. Gursev, F. Iachello, *Ann. Phys.* 148 (1983) 346.
- [35] R. Feynman, A. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

- [36] L.S. Schulman, *Techniques and Applications of Path Integration*, Wiley, New York, 1981.
- [37] I.H. Duru, H. Kleinert, *Phys. Lett. B* 84 (1979) 185.
- [38] N.K. Pak, I. Sokmen, *Phys. Rev. A* 30 (1984) 1629.
- [39] M. Eshghi, S. Arbabi Moghaddam, *Math. Methods Appl. Sci.* 38 (2015) 5124.
- [40] W.C. Qiang, S.H. Dong, *Europhys. Lett.* 89 (2010) 10003.
- [41] F.A. Serrano, X.Y. Gu, S.H. Dong, *J. Math. Phys.* 51 (2010) 082103.
- [42] S.H. Dong, M. Cruz-Irisson, *J. Math. Chem.* 50 (2012) 881.
- [43] F.A. Serrano, M. Cruz-Irisson, S.H. Dong, *Ann. Phys.* 523 (2011) 771.
- [44] R. De, R. Dutt, U. Sukhatme, *J. Phys. A: Math. Gen.* 25 (1992) L843.
- [45] R. Dutt, A. Khare, Y.P. Varshni, *J. Phys. A* 28 (1995) L107.
- [46] C.S. Jia, Y.F. Diao, M. Li, Q.B. Yang, L.T. Sun, R.Y. Huang, *J. Phys. A* 37 (2004) 11275.
- [47] A.D. Alhaidari, *Phys. Rev. Lett.* 87 (2001) 210405.
- [48] E. Schrödinger, *Ann. Phys.* 384 (1926) 361.
- [49] M.J. Englefield, *J. Aust. Math. Soc.* 8 (1968) 557.
- [50] R.A. Swainson, G.W.F. Drake, *J. Phys. A: Math. Gen.* 24 (1991) 79.
- [51] Y. Ran, L. Xue, S. Hu, R.K. Su, *J. Phys. A: Math. Gen.* 33 (2000) 9265.
- [52] G. Chen, *Phys. Lett. A* 326 (2004) 55.
- [53] C. Gang, *Chin. J. Phys.* 14 (2005) 1075.
- [54] A. Arda, R. Sever, *J. Math. Chem.* 50 (2012) 971.
- [55] D.R.M. Pimentel, A.S. de Castro, *Eur. J. Phys.* 34 (2013) 199.
- [56] M. Eshghi, *Can. J. Phys.* 91 (2013) 71.
- [57] P. Grave, C. Gauthier, *Amer. J. Phys.* 79 (2011) 447.
- [58] E. Olgar, R. Koc, H. Tutunculer, *Chin. Phys. Lett.* 23 (3) (2006) 539.
- [59] C.F. Hou, X.D. Sun, Z.X. Zhou, Y. Li, *Acta Phys. Sin.* 48 (3) (1999) 385.
- [60] C.Y. Chen, C.L. Liu, F.L. Lu, D.S. Sun, *Acta Phys. Sin.* 52 (2003) 1579.
- [61] Y. He, Z.Q. Cao, Q.S. Shen, *Phys. Lett. A* 326 (5–6) (2004) 315.
- [62] G. Chen, Z.D. Chen, Z.M. Lou, *Phys. Lett. A* 331 (6) (2004) 374.
- [63] W.C. Qiang, *Chin. J. Phys.* 12 (10) (2003) 1054.
- [64] L.Z. Yi, Y.F. Diao, J.Y. Liu, C.S. Jia, *Phys. Lett. A* 333 (3–4) (2004) 212.
- [65] C.S. Jia, Y. Li, Y. Sun, J.Y. Liu, L.T. Sun, *Phys. Lett. A* 311 (2–3) (2003) 115.
- [66] F. Domínguez-Adame, *Phys. Lett. A* 136 (4–5) (1989) 175.
- [67] Z.Q. Ma, S.H. Dong, X.Y. Gu, J.A. Yu, M. LozadaCassou, *Internat. J. Modern Phys. E* 13 (3) (2004) 597.
- [68] C.Y. Chen, D.S. Sun, F.L. Lu, *Acta Phys. Sin.* 55 (8) (2006) 3875.
- [69] W.C. Qiang, S.H. Dong, *Phys. Lett. A* 372 (27–28) (2008) 4789.
- [70] S.S. Dong, S.H. Dong, H. Bahlouli, V.-B. Bezerra, *Internat. J. Modern Phys. E* 20 (2011) 55.
- [71] S.M. Ikhdaïr, R. Sever, *Ann. der Phys.* 16 (3) (2007) 218.
- [72] X.Y. Chen, T. Chen, C.S. Jia, *Eur. Phys. J. Plus* 129 (2014) 75.
- [73] C.S. Jia, J.W. Dai, L.H. Zhang, J.Y. Liu, G.D. Zhang, *Chem. Phys. Lett.* 619 (2015) 54.
- [74] H.B. Liu, L.Z. Yi, C.S. Jia, *J. Math. Chem.* 56 (2018) 2982.
- [75] A.D. Polyanin, A.V. Manzhirov, *Handbook of Integral Equations*, CRC press, New York, Wash-Ington, 1998.
- [76] S. Ortakaya, *Chin. Phys. B* 21 (2012) 070303.
- [77] M. Zarezadeh, M.K. Tavassoly, *Chin. Phys. C* 37 (2013) 043106.
- [78] M.R. Spiegel, *Schaum's Outline of Theory and Problems of Laplace Transforms*, Schaum Publishing Co., 1965.
- [79] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformations*, Springer, New York, 1974.
- [80] T. Dos, A. Arda, 2014, arXiv:13085295v2 [math-ph].
- [81] R. Kumar, F. Chand, *Phys. Scr.* 85 (2012) 055008.
- [82] Chun-Sheng Jia, Chao-Wen Wang, Lie-Hui Zhang, Xiao-Long Peng, H.M. Tang, Ran Zeng, *Chem. Eng. Sci.* 183 (2018) 26.
- [83] C.S. Jia, R. Zeng, X.L. Peng, L.H. Zhang, Y.L. Zhao, *Chem. Eng. Sci.* 190 (2018) 1.
- [84] X.L. Peng, R. Jiang, C.S. Jia, L.H. Zhang, Y.L. Zhao, *Chem. Eng. Sci.* 190 (2018) 122.
- [85] C.S. Jia, L.H. Zhang, X.L. Peng, J.X. Luo, Y.L. Zhao, J.Y. Liu, J.J. Guo, L.D. Tang, *Chem. Eng. Sci.* 202 (2019) 70.
- [86] C.S. Jia, C.W. Wang, L.H. Zhang, X.L. Peng, X.L. Peng, R. Zeng, X.T. You, *Phys. Lett.* 676 (2017) 150.
- [87] M.H. Pacheco, R.R. Landim, C.A.S. Almeida, *Phys. Lett. A* 311 (2003) 93.
- [88] M.H. Pacheco, R.V. Maluf, C.A.S. Almeida, R.R. Landim, *Europhys. Lett.* 311 (2014) 10005.