



Grade 10 Students' Technology-based Exploration Processes of Narratives Associated with the sine Function

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Abstract

Researchers point out that more educational research is needed to study students' understanding of trigonometric topics. The present research attempts to study a group of three high-achieving eleventh grade students' realization of trigonometric words and narratives associated with the sine function. The learning of the students was video recorded and analyzed using the commognitive theoretical framework. The research results indicated that the students performed inductive and deductive processes, with the mediation of technology, to realize new trigonometric significations; the unit circle and the coordinate system, including words and narratives associated with these significations. Technology functioned as static and dynamic visual mediators. It functioned as a static visual mediator when it mediated the deductive processes of the students, while it functioned as a dynamic visual mediator when it mediated the inductive processes of the students. The students' processes, technology and the teacher's processes combined to mediate the students' sameness, encapsulation and reification of trigonometric words and narratives. The research results indicate the important role of the different trigonometric significations for students' understanding of trigonometric functions. These significations were mediated by technological tools.

Keywords: trigonometry, students' routines, commognition, exploration processes, narratives, the sine function

INTRODUCTION

Despite the acknowledged difficulties in students' learning of trigonometric functions (Demir, 2012), research on trigonometry learning is sparse and quite limited (Nejad, 2016; Weber, 2005). Only few studies appear to analyze students' knowledge construction of trigonometric functions (See for example Brown, 2005). This could be claimed also for students' understanding of trigonometric functions in a technological environment, though it attracted in the last decade the attention of researchers (e.g., DeJarnette, 2014; Demir, 2012; Moore, 2009; Ross, Bruce, & Sibbald, 2011).

Researchers who were interested in the role of technology in the learning of trigonometry pointed at its contribution to students' learning of trigonometric concepts and relations through connecting different trigonometric representations. Blackett and Tall (1991)

pointed out that the main contribution of triangle-trigonometry software to trigonometry learning was its contribution to the exploration of relationships between visual and numeric representations of trigonometric ratios as they appear in right-angled triangles. In particular, Demir (2012) found that GeoGebra can facilitate students' connections between the three contexts of trigonometric functions: the right-angled triangle, the unit circle and the function graph. Kissane and Kemp (2009) explored the potential of technology, specifically the graphic calculator, to help students make connections between trigonometric and circular functions. They reported that technology facilitated students' exploration of narratives related to the trigonometric graphs, as those related to their periodicity, amplitude, their maximum and minimum point(s) and their zeroes, in addition to those related to trigonometric identities and equations. This potential of

Contribution to the literature

- Researchers point out that despite the difficulties in students' learning of trigonometric functions, educational research in this field is sparse.
- This is also the case with the educational research regarding learning trigonometry with technology.
- The present paper attempts to contribute to the understanding of students' learning of trigonometry, specifically when technology is utilized.
- The paper uses the Commognition framework to analyze students' processes of exploring the characteristics of the sine function and the relationships between its different significations.

technology to facilitate the mathematical connections was also reported by Wilson (2008) who reported that dynamic web tools facilitated students' interaction with the unit circle to connect the graph and the algebraic expression of trigonometric functions. In addition, the web tools facilitated the learning of difficult topics as the exploration of Fourier series, as well as the complex exponential functions and their periodicity. In the present research, we study secondary students' learning of trigonometric realizations and narratives, as $\sin \alpha = 0$, using technology, specifically GeoGebra. Doing that, we utilize the lens of the commognitive framework (Sfard, 2007, 2008). This framework enables us to address the significations of the three trigonometric functions and the transition of the students from one signification to another as realization of the former in the latter.

The Commognitive Framework as a Lens to Understand Students' Realizations of Trigonometric Entities

The commognitive framework considers learning mathematics as adjusting and extending the participant's discourse in mathematics through communication, written or verbal (Sfard, 2007). Specifically, the commognitive framework tries to take account of three aspects of learning: (1) The object of learning (what change was expected to occur as a consequence of learning?); (2) the learning process (How did the participants as students and the teacher work toward the change?); and (3) the learning outcome (Has the expected change occurred?). Moreover, the interpretive commognitive framework is based on the assumption that "thinking is a form of communication and that learning mathematics is tantamount to modifying and extending one's discourse" (Sfard, 2007, p. 567). Furthermore, discursive change, which is the essence of learning, is prompted by commognitive conflict arising in a mathematical situation whenever different interlocutors act according to different discursive rules.

Sfard describes four characteristics of the mathematical discourse that help analyze students' learning of mathematics (Sfard, 2007, p. 572-575): word use, visual mediators, narratives and routines. Mathematical words are the means by which the participants in a mathematical discourse express

mathematical ideas, or/and to communicate with the other participants regarding these ideas. In such a discourse, a learner studies new uses of previously-met mathematical words, or/and learns new mathematical words that he or she has never used before. For example, children participating in a mathematical discourse about triangles may change their conceptions of what a triangle is as a result of working in a dynamic geometric environment (Sinclair & Moss, 2012).

Visual mediators are visual objects and resources utilized by participants in a mathematical discourse to identify mathematical ideas and coordinate their learning communication. These mediators include symbols such as numerals, algebraic letters and representational entities such as tables, graphs and diagrams. These mediators are utilized for thinking or communicating in a mathematical discourse (Sfard, 2008). Furthermore, GeoGebra provides a context where it is easy to produce visual mediators by drawing graphs of various functions (Berger, 2013).

Narratives are texts, whether spoken or written, that describe mathematical objects, or relations between these objects, and that could be evaluated by the participants in the mathematical discourse. Examples of narratives are definitions, equations, or theorems. Berger (2013) says that within technology-based mathematical learning, mathematical narratives are endorsed as correct if they agree with the traditional mathematical narratives.

Routines are repetitive patterns of the participants' actions and communications, characteristic of a specific discourse. These routines characterize the use of mathematical words, the use of visual mediators or the processes of creation, substantiation or development of mathematical narratives. Examples on mathematical routines are methods of arithmetic calculations and of mathematical proof. Sfard (2008) divides routines into *explorations* whose aim is to advance discourse through the production, development or verification of endorsable narratives (whether a mathematical conjecture or a mathematical relation); *deeds* whose aim is to change the actual objects, whether physical or discursive; and *rituals* whose aim is to create and sustain social approval with other participants in the mathematical discourse. This is usually done through aligning the mathematical activity of the participants

with other participants' routines. Moreover, rituals could involve imitating routines of other participants in the mathematical discourse (Berger, 2013). Sfard suggested to divide explorations into three types: construction, substantiation and recall. *Construction* aims at creating new endorsed narratives, *substantiation* aims to decide whether to endorse previously created narratives, while *recall* aims to call upon narratives endorsed in the past.

In their exploration of words and narratives, students use what Sfard (2008) calls 'saming', reification and encapsulation. Saming is linked with the process of associating a term with several mathematical objects that look different. It could be applied to discursive objects that are all realisations of the same signifier and can be part of the learner's construction of new mathematical objects. The necessary basis for such saming is the fact that whatever is said with the common signifier (e.g., basic quadratic function) and turns out to be endorsable when translated into a narrative about any of this signifier's realizations (the parabola) will be endorsable also when translated into a narrative about the other realization (the expression x^2). Reifying occurs when the participants in the discourse turn processes into object, which is the beginning of objectification, and, if completed, will leave us with the "objective" existence of the object-like referent. Encapsulation occurs when the participants in the discourse assign a noun or pronoun (signifier) to a set of objects, so that the narratives about the members of this set that have, until now, been told in plural could now be told in the singular.

The rules of narrative construction include three meta-discursive manipulations, known as deduction, induction, and abduction. "Deduction takes place when a new narrative is obtained from previously endorsed narratives with the help of well-defined inferring operations" (Sfard, 2008, p. 229). The basic form of such operation is: If you already endorsed the narratives $P \rightarrow Q$ and P , then Q can be endorsed as well. "Induction is a process in which a new narrative on any object is obtained from a finite number of already endorsed narratives on specific instances of this object (p. 229). Abduction is a process in which endorsability of a new narrative is a result of the endorsement of its necessary consequence (p. 229).

In addition to the above, Sfard makes use of the term 'realizations of a signifier'. Describing this term (p. 154), she says that "mathematical communication involves incessant transitions from signifiers to other entities that, from now on, will be called realizations of the signifiers. Signifiers are words or symbols that function as nouns in utterances of discourse participants, whereas the term realization of a signifier S refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S ". In addition, realizations, according to Sfard, take the form of concrete objects, drawings, algebraic symbols,

written or spoken words, or gestures. Moreover, she emphasizes that the signifier-signified relation is symmetrical (p. 155).

The commognitive framework was used by various researchers to describe and analyze students' learning (e.g., Berger, 2013; Swidan & Daher, 2019) of and teachers' instruction (e.g., Nardi, Ryve, Stadler & Viirman, 2014; Viirman, 2012) of different mathematical concepts and relationship, when the emphasis of the analysis was on words, visual mediators, narratives and routines. Researchers also used the commognitive framework to study the relation of teachers' instruction with students' learning (e.g., Kotsopoulos, Lee, Heide & Schell, 2009), and to analyze instructional materials (e.g., Newton, 2009). In addition, researchers used the commognitive framework to analyze the social aspect of mathematical learning including students' identity (e.g., Heyd-Metzuyanin & Graven, 2016).

In more detail, Berger (2013) and Pettersson, Stadler and Tambour (2013) used the commognitive framework to study students' learning of the function concept. Doing so, they explained the success of the participants in their learning of the function concepts by looking at the properties of their commognitive activity, for example their routines (Berger, 2013) or their use of the visual mediators (Pettersson et al., 2013). More specifically, Berger (2013) results indicated that one of two participants, who was not evidently successful in deciding if a vertical asymptote is associated with an undefined point, was involved in routines of ritual type through imitations. Pettersson et al. (2013) reported that two of four participants, who had a transformed understanding of the concept of function, expanded their use of mathematical words and developed their narratives from everyday examples and the concept of the function as a rule to the function as pairs. Furthermore, the visual mediators were not critical to their understanding. At the same time, the other two participants, who did not have a transformed understanding of the concept of function, used few mathematical words, did not have routines sufficient to decide whether a curve represents a function or not, and their narratives were strongly connected to visual mediators.

Researchers also studies teachers' instruction using the commognitive framework. Viirman (2012), for example, used this framework to analyze teachers' instructional routines. More specifically, Viirman (2012) analyzed the teaching of the topic of functions by seven teachers in three Swedish universities. The teachers' discursive practices were found to contain two intertwined practices: mathematical discourse and the discourse of mathematical teaching. Routines specific for mathematical discourse were construction and substantiation routines, while the didactical routines included motivation, explanation, activation and recall.

RESEARCH RATIONALE AND GOALS

Demir and Heck (2013) describe trigonometry as an important subject in secondary mathematics education and beyond, where the curriculum of school trigonometry is distributed over several school years. This curriculum involves the introduction of sine, cosine and tangent as functions of an angle, either through utilizing right-angled triangles or the unit circle, or as functions of a real number. Weber (2005) points out that despite the reported difficulties with learning trigonometric functions, the related educational research literature is scant. Kissane and Kemp (2009) make a similar comment regarding the little research on the use of technology in trigonometry learning, where some of this research has been involved mainly with suggesting methods and activities to integrate technology in trigonometry learning (e.g., Kissane & Kemp, 2009; Wilson, 2008).

In addition to the above, investigating students' learning of trigonometry was performed mainly through surveys and interviews (e.g., Weber, 2005; Kepceoglu & Yavuz, 2016). Weber (2005), for example, interviewed the participants regarding their conceptions of trigonometric functions, using questions such as "When is $\sin\theta$ decreasing and why?" On the other hand, Demir (2012) distributed worksheets, tests and held interviews to analyze students' conceptions of trigonometric functions. The tools also included observations of students' discussions how to solve the questions in the worksheets. Demir's study showed learning processes in order to explain students' conceptions of a trigonometric products. The learning processes were described shortly, as, for example, the processes that explained students' conceptions of the sine values as graphically signified. This understanding was realized through connecting the sine values with the periodicity of the function, by the idea of copy and paste (p. 101). The present research uses observations to examine processes of students' actual learning of trigonometry. Thus the focus here is on the trigonometric processes, where the products are also analyzed, but in terms of the related processes. This analysis of the processes is done through the lenses of the commognitive framework. The use of this theory fits the purpose of the current research to analyze learning processes of trigonometric function, for one of the main focuses of the commognitive framework is students' routines that include processes of understanding. In addition, the students' realizations of a former realization in a later one is studied, which also would enable to show the difficulties of students in performing such realizations.

Research on the role of technology in mathematics learning, as described above, points at the advantages of this role. On the other hand, DeJarnette (2014), utilizing quantitative analysis of students' pre- and post-tests, found that many of the differences in students' scores in

the topic of trigonometric functions, due to technology use, were not statistically significant. The inconsistency in past research findings regarding the impact of technology on students' learning of geometric concepts points at the need for further research that examines this impact. Both qualitative and quantitative researches are requested in this field. The present research attempts to contribute to the qualitative research regarding students' understanding of trigonometric functions by utilizing the commognitive framework of Sfard (2007, 2008). Specifically, we describe grade 10 students' exploration of the sine function in the unit circle context after they had explored this function in the right-angled triangle context. Doing that, they made connections between the two contexts to conceptualize the realization of one context's signification in the other signification. This realization was accomplished by utilizing technology, in this case GeoGebra, which helped the students arrive at their conceptualizations.

METHODOLOGY

Research Setting and Participants

The research was conducted in Grade eleven mathematics class that studied trigonometry as part of the mathematics program. The reported participants were three eleventh grade students, aged 16-17 years, with the fictive names: Ayat, Adan and Saba. The three students were described by their teacher as very good students in mathematics (with grades between 90 and 95 in mathematics at the first trimester of the academic year 2016-2017). The choice of the students was based on convenience sampling; i.e. it depended on the possibility or easiness with which a researcher could get in touch with the participants. The mathematics class was chosen due to the consent of the teacher and students to be part of a research that investigates the class learning of the concept of trigonometric function. The students whose learning is reported participated at their own will and expressed their consent to be part of the research.

The students learned as a group, in the second trimester of the academic year 2016-2017 to explore different properties of the sine function in two new significations: the unit circle and the coordinate system.

Data Collection

The computer screens of the group members, as well as their work on the computer, were video-recorded by utilizing a computer program that captured the footage in two different windows, one window for the student and the second window for the computer screen. The main role of the teacher in conducting the learning activity was to ask questions. To answer the research question, we analyzed three lessons in which the students performed a sequence of activities related to the

concept of the sine function. Each lesson lasted for 45 minutes.

Data Analysis

The data analysis involved repeatedly watching the videos, transcribing them, and reading the transcripts. Analyzing the transcripts was done through focusing on the students' routines and associated realization processes. We identified the realization processes as processes that involves being engaged with the realizations of a signifier. Examples on realizations of a signifiers are (Sfard, 2009, p. 154): 'Table of values' is a realization of the signifier 'function g' ', '5' is a realization of the signifier 'slope g' ', 'The x-coordinate of the intersection of the two straight lines that realize $7x + 4$ and $5x + 8$, respectively' is a realization of the signifier 'The solution of the equation $7x + 4 = 5x + 8'$.

We categorized the routines taking into consideration the types suggested in Sfard (2008). A routine was considered an exploration when the participant performed the routine in order to establish a narrative. A routine was considered an exploration of the type 'construction', when the participant performed the routine in order to endorse a narrative or to verify a mathematical relationship that was previously conjectured or arrived at. Furthermore, we considered routines to be rituals when the participant performed the routine for a social concern; i.e. was a way of getting attention and approval of others and becoming a part of a social group. We considered a routine to be a deed, when the routine produces a change in environment. Sfard (2008) argues that mathematicists' concern about deeds should be the starting point for any discursive development as the explorative routines (p. 245). In addition, Sfard (2008) argued that could begin their life for children as neither deeds nor explorations but rituals, where the primary goal of the discursive actions is creating and sustaining a bond with other people (p. 241). This beginning could turn into exploration actions in due time. We argue that this situation satisfies not only children's routines but routines of all students.

Furthermore, we identified the rules of narrative construction into the two meta-discursive manipulations, deduction and induction according to Sfard's (2008) description, where deduction occurs when a new narrative is obtained from previously endorsed narratives through well-defined inferring operations. Deduction takes the form: If you already endorsed the narratives $P \rightarrow Q$ and P , then Q can be endorsed as well. In addition, induction occurs when a new narrative on an object is obtained from a finite number of already endorsed narratives on specific instances of this object. An example on deduction is the deduction of relations associated with the sine function in the unit circle depending on these relations in the right-angled triangle. Transcript Ex.1 shows such a deduction process.

- | | |
|------|--|
| Saba | [Saba manipulated the angle till it got 30°] We know that the sine is the opposite divided by the hypotenuse. |
| Adan | Being in the unit circle, this means that the length of the hypotenuse is one, so to find $\sin 30^\circ$ we need only take care of the length of the opposite side, because dividing by 1 will not change anything. |

Transcript Ex. 1: A deduction process

An example on induction is the work of the students with a technological tool to arrive at different realizations of a signifier in order to arrive at an appropriate trigonometric narrative. Transcript Ex.2 shows such an induction process.

- | | |
|---------|--|
| Saba | O.K, let us drag the point B and see which angles give us an opposite side whose length is equal to 0. |
| Adan | [dragged the point B] Look, if the angle is 360 , then it satisfies $\sin \alpha = 0$. |
| Teacher | Excellent. This is the first point that satisfies. See if there are more. |
| Ayat | I think that if the angle is 90 , it also will satisfy $\sin \alpha = 0$. |

Transcript Ex. 2: An induction process

In addition, we identified sameness as realizing a signifier in two or more different ways, or two or more different contexts (Sfard, 2008, p. 189). We identified reification as occurring in the processes of discursive compression, so it has a 'compacting' effect (Sfard, 2008, p. 120). At the same time, we identified encapsulation as occurring in the processes of assigning a noun or pronoun (signifier) to a specific set of objects, so that some of the stories about the members of this set can be told in singular instead of plural (Sfard, 2008, p. 171). An example of the sameness process is the sameness of the realizations $\sin 0=0$, $\sin 360=0$ and $\sin 180=0$ as realizations of the same signifier of trigonometric narratives; i.e. $\sin \alpha = 0$. This sameness led into the encapsulation of these realizations into "the sin of 180 or its multiples equals 0 ". Here, the description uses the singular noun instead of talking about different realizations as plural. Reification of these narratives resulted in " $\sin 180n = 0$, for any integer n ". Here, we are talking about one object, namely $\sin 180n$.

Learning Material

Following are the two activities with which the group of students was engaged to explore and substantiate narratives related to the two new significations of the trigonometric functions.

Activity 1

Using the accompanying applet of the unit circle, answer the following questions:

1. Find an angle α which satisfies $\sin \alpha = 0$. How many angles satisfy this equation?
2. Find an angle α which satisfies $\sin \alpha = 1$. How many angles satisfy this equation?
3. Find an angle α which satisfies $\sin \alpha = -1$. How many angles satisfy this equation?
4. Explore the relationship between $\sin \alpha$ and $\sin(180 - \alpha)$.
5. Explore the relationship between $\sin(\alpha)$ and $\sin(180 + \alpha)$.

Activity 2

1. Draw the function $f(x)=\sin \alpha$ in GeoGebra.
2. Explore the graph of this function regarding: intersection points with the x-axis, intersection points with the y-axis, domains in which the function is increasing or decreasing, domains in which the function is positive or negative, maxima or minima points.
3. What can you say about the behavior of the function $f(x)=\sin \alpha$? What do you think the reasons are behind this behavior?

RESULTS

The goal of the first activity was to engage the students with the unit circle signification in order to study the trigonometric functions in new domains; i.e. for angles more than 360 degrees. Being engaged with the unit circle signification, the students endorsed several narratives related to $\sin \alpha$. We report here the sequence of routines through which the students came to realize the unit circle signification of the trigonometric functions, and then report the sequence of routines through which they endorsed narratives related to the sine function, as $\sin 180k = 0$ and $\sin(360k + 90) = 1$. Afterwards, we will report the sequence of routines that the students utilized to realize the coordinate system signification of the trigonometric functions and thus the period and the periodicity of the trigonometric function.

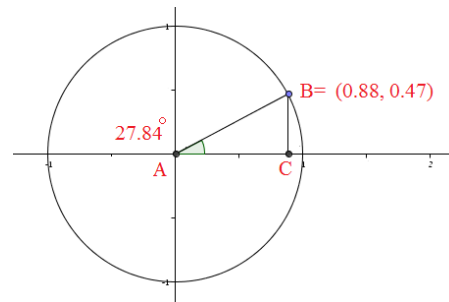
Exploring a New Signification of the Trigonometric Functions: The Unit-circle

The teacher described the lesson goal, requesting the students to explore the properties of the trigonometric function $\sin x$. This request started the students' exploration of the new signification of the trigonometric functions.

- 1 Teacher Today, we will use the unit circle to study properties of the trigonometric

function $\sin x$. Manipulate the angle in the applet. What do you notice?

- 2 Students The students began to manipulate the angle.



- 3 Ayat O.K. it is an angle in a right-angled triangle.

- 4 Adan It is part of a circle. [...]. Why is it in a circle?

- 5 Teacher How can you characterize this circle?

- 6 Adan Its center is in (0,0). [...]. its radius is one.

- 7 Teacher We call it a unit circle. What is the value of sine 30° and cosine 30°?

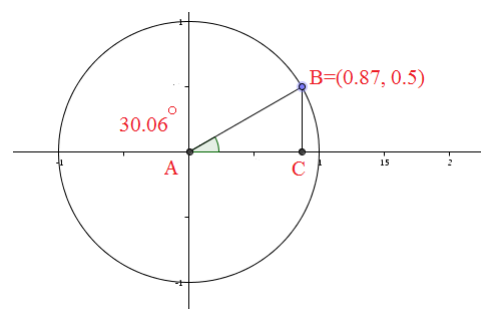
- 8 Saba [Saba manipulated the angle till it got 30°] We know that the sine is the opposite divided by the hypotenuse.

- 9 Adan Being in the unit circle, this means that the length of the hypotenuse is one, so to find sine 30° we need only take care of the length of the opposite side, because dividing by 1 will not change anything.

- 10 Saba This is the y-coordinate of point B, because the center of the circle is in (0,0).

- 11 Ayat The cosine is the x-coordinate of point B.

- 12 Saba sine 30° is point five, and cosine 30° is point eighty seven.



13 Ayat Yes, fantastic. We got these values of the sine and cosine, when we computed them with the calculator. Right Saba?

14 Saba [Nodded with her head]

Transcript 1: Exploring the new unit-circle signification of trigonometric function

The students started their exploration by manipulating the angle in the unit circle. These processes began as rituals and deeds at the same time for they were performed as consequence of the teacher's request [R1] and, at the same time, they entailed changing mathematical objects, in our case the given angle [R2].

Adan identified, by engaging in deductive processes [R8-R11], the unit-circle context as realization of the 'right-angled triangle' signification of trigonometric functions. The students' engagement with the deductive processes led them to construct two sub-narratives that were related to the definition of the sine and cosine in a unit circle [R10, R11]. In addition, these narratives (i.e., definitions) led the students to realize sine 30° and cosine 30° in the unit-circle signification [R12]. The technology here functioned mainly as a visual mediator that mediated students' deductive reasoning. It also functioned as dynamic mediator that facilitated the transition from a specific angle to another specific angle.

Endorsing the Narrative $\sin 180n=0$

The next task for the students was to explore the realizations of $\sin \alpha = 0$ in order to endorse the narrative $\sin 180n = 0$. To do so, the students went through a sequence of routines: Endorsing sub-narratives needed for the realization of $\sin \alpha = 0$, exploring realizations of $\sin \alpha = 0$, reifying angles more than 360, and endorsing the narrative $\sin 180n = 0$. We describe each of these routines below.

Endorsing Sub-narratives Needed for the Realization of the Narrative $\sin \alpha = 0$

As a result of the teacher's request, the students started to explore the conditions for satisfying the narrative $\sin \alpha = 0$ in the unit circle signification.

15 Teacher Let us use the unit circle to study when $\sin \alpha = 0$.

16 Ayat Right, this is the first question. It requests to find the angles that satisfy $\sin \alpha = 0$.

17 Teacher We want to use the applet to find these angles.

18 Ayat It means I should look at the length of BC, so to get $\sin \alpha = 0$, the length of BC should be zero.

Transcript 2: Unit-circle signification of the sub-narratives needed for realizing the narrative $\sin \alpha = 0$.

In Transcript 2, the students were engaged again with deductive processes to explore the sub-narratives that need to be endorsed in order to realize the narrative $\sin \alpha = 0$ [R15-R18]. This was done by deductive reasoning regarding the conditions for realizing $\sin \alpha = 0$ in the new signification, i.e. the unit circle signification. The deductive processes led the students to construct the sub-narrative: "the opposite side of the angle should be zero" [R18]. The routine performed by the students could be described as an exploration routine mediated by deductive processes. The students were aware of the questions in the activity [R16] and utilized the technology as a visual mediator mediating their deductive processes.

Exploring realizations of the narrative $\sin \alpha = 0$

The deductive processes first led to the sub-narrative that the students utilized to explore numeric realizations of the original narrative $\sin \alpha = 0$. This exploration of the numeric realizations was carried out using inductive meta-discursive processes escalated by students' work with the applet [R19-R21].

19 Saba O.K, let us drag the point B and see which angles give us an opposite side whose length is equal to 0.

20 Adan [dragged the point B] Look, if the angle is 360, then it satisfies $\sin \alpha = 0$.

21 Teacher Excellent. This is the first point that satisfies. See if there are more.

22 Ayat I think that if the angle is 90, it also will satisfy $\sin \alpha = 0$.

23 Adan Let us see [She drags the point B till she gets the mentioned angle] Ayat, you are incorrect, because the opposite side is 1, not 0. So, you cannot say that $\sin (90) = 0$.

24 Ayat O.K, let us drag and see when we get an opposite side whose length is equal to 0.

25 Adan [drag the point B on the unit circle]. I found it. When $\alpha = 180$, we have $\sin \alpha = 0$. We can say $\sin 180 = 0$.

26 Teacher Can you explain for the other group members why this is correct?

27 Adan See, the length of BC which is the opposite ... it has no length, so its zero, so this is correct.

28 Saba This means that we got till now two angles that satisfy $\sin \alpha = 0$, when $\alpha = 360$ and when $\alpha = 180$.

Transcript 3: Exploring realizations of the narrative $\sin \alpha = 0$

The inductive processes, supported by the applet as a dynamic mediator, helped the students arrive at one realization of the given narrative $\sin \alpha = 0$; i.e. $\alpha = 360$ [R20]. Combining between the deductive [R20, R27] and the inductive [R25, R28] processes, and working with the applet, the students arrived at a second realization of the narrative $\sin \alpha = 0$, which is $\alpha = 180$ [R25, R27]. These realizations were first step towards saming the two realizations as signifying $\sin \alpha = 0$.

The applet, as a dynamic mediator, also supported the students to substantiate another possible realization of the narrative; $\sin \alpha = 0$; $\alpha = 90$, which was suggested by Ayat. Manipulating the angle inside the unit circle [R23], Adan did not agree to endorse the new realization of the original narrative suggested by Ayat.

Reifying Angles more than 360

Led by the teacher's suggestion to find the value of $\sin 540$ [R32], the students performed a ritual routine, to comply with the teacher's request, but soon this routine turned into an exploration routine in which the students explored new realizations of the trigonometric narrative $\sin \alpha = 0$. Doing so, they came to reify two trigonometric objects: 540 as a signifier of the angle object, and $\sin 540$ as a signifier of the $\sin \alpha$ object.

29 Teacher Try to see if there are other angles in the unit circle with $\sin \alpha = 0$.

30 Adan Nothing, because we arrive at 360 which is the last thing in the circle, nothing after that.

31 Ayat So, we found two angles.

32 Teacher O.K. Let us find the value of $\sin 540$.

33 Ayat Didn't we say that the last point in the circle is 360, so how can we find $\sin 540$?

34 Teacher Why don't you discuss this issue with your mates?

35 Ayat What do you think Adan and Saba?

36 Saba Let us think a little.

37 Students [The three students looked at the unit circle with astonishment. Adan started to increase the angle in the unit circle, stopped when the angle measure was

360, but after one second she continued to increase the angle]

38 Adan I have an idea. I don't know whether it is right or not.

39 Adan We know that one turn is 360 degrees. So, if I want to find the sine of an angle that is greater than 360, I am in the second turn, which means that to arrive at 540 I need to rotate one turn, 360, and continue the rest.

40 Saba I think your talk is logical, so the unit circle is not only till 360 as we thought before. We can arrive at any number we want by doing turns; first turn, second turn, third turn, etc.

41 Ayat Assuming I understood well, you say that 540 is in the second turn, because we walk 360, and then continue 180, thus arriving at 540.

42 Saba On the other hand, we know that $\sin(180) = 0$, which means that $\sin(540) = 0$.

Transcript 4: Reification of new mathematical objects

The students started to use new mathematical words, although this use was not yet precise. One of these words was the 'turn', where the students talked about a first turn, a second turn, a third turn [R39-R41]. They started to use this new mathematical word as part of their exploration of a new discursive object that was mediated by the teacher's suggestion and their work with the applet as a visual mediator. This new discursive object was the angle in its rotation signification [R39-R41]. In their exploration of the new discursive object, the students at the beginning used daily terms/phrases as "continue the rest" [R27] instead of "continue the rotation", "arrive at any number we want" [R28], instead of "arrive at the angle we want" or "we walk 360" [R41] instead of "we rotate 360".

In addition to development of the use of the mathematical word for the discursive mathematical object, the new object was realized first as a process, through Adan's increase of the angle in the unit circle. The teacher's request to find $\sin 540$ helped the students to reify this process into an object, for they needed to find its 'sine value' and thus to realize 540 as a signifier of the angle object.

Endorsing the narrative $\sin 180k = 0$

Exploring rotations more than 360 as signifiers for the angle object, helped the students endorse all the realizations of the narrative $\sin \alpha = 0$. This time, the

applet helped the students in their deductive processes regarding two trigonometric narratives ($\sin \alpha = 0$ and $\sin 180k = 0$) and their discursive meanings.

- 43 Teacher So, what do we conclude?
- 44 Ayat We conclude that for every 180 degrees we get an angle with $\sin \alpha = 0$, and we have endless points for we have endless turns.
- 45 Teacher Can you write that in letters?
- 46 Saba $\sin \alpha = 0$ when α is 180 or multiple of 180.
- 47 Teacher In letters.
- 48 Ayat $\sin 180k = 0$ when k is a positive integer.

Transcript 5: Endorsing the narrative $\sin 180k = 0$

In spite of their daily use of mathematical words, the students endorsed successfully the narrative related to the numeric realizations of the angle that satisfies $\sin \alpha = 0$. This endorsement was performed deductively, when Saba named $\sin(540)$ with $\sin(180)$ [R42], and Ayat encapsulated the two realizations of $\sin \alpha = 0$ into a general narrative [R44]. The naming and afterwards the encapsulation were mediated by the teacher's directions and the applet as a dynamic mediator. This encapsulation advanced gradually, first with daily writing [R44], afterwards with mathematical writing [R46] and finally with mathematical symbols [R48] to become a reification for the mathematical narrative.

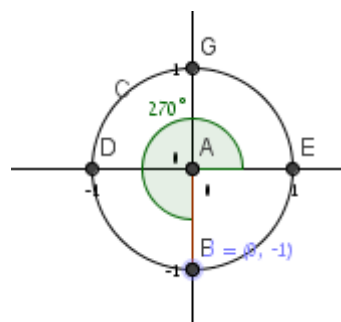
Endorsing the narrative $\sin(360k + 90) = 1$

The next task for the students was to explore when $\sin \alpha = 1$ in order to endorse the narrative $\sin(360k + 90) = 1$. To endorse this narrative, the students went through a similar sequence of routines as in the case of endorsing $\sin 180k = 0$. This sequence was related to realizing the narrative $\sin \alpha = 1$ and included constructing the following sub-narratives: $\sin 90 = 1$, $\sin 270 = 1$ (a wrong narrative), "the sine value is not related only to the length of the opposite edge but also to its location", $\sin 270 = -1$, $\sin 450 = 1$, $\sin 810 = 1$. The construction of the previous sub-narratives led the students to constructing and substantiating the narrative $\sin(360k + 90) = 1$.

Transcript (6) describes students' construction of the sub-narrative $\sin 90 = 1$ and wrongly the sub-narrative $\sin 270 = 1$. This construction was initiated by the teacher's request to find the angles that satisfy $\sin \alpha = 1$.

- 49 Teacher The next question requests us to find the angles that satisfy the narrative $\sin \alpha = 1$.

- 50 Adan This question is similar to the previous one, but here we need to look for angles that satisfy $BC = 1$.
- 51 Ayat You are right. Let us move point B and watch.
- 52 Saba [Drags point B and stops when getting a right angle] See, when the angle is ninety, BC is one.
- 53 Adan This means that $\sin 90 = 1$.
- 54 Teacher This is the first angle that satisfies $\sin \alpha = 1$. Look please for other angles.
- 55 Ayat [Drags point B and stops when getting the angle 270]. $\sin 270$ is one.
- 56 Teacher I tell you that it is not 1, but -1. Can you tell me why?
- 57 Adan Why teacher? The sine is the opposite divided by the hypotenuse.



- 58 Ayat Maybe it has to do with the place of the opposite edge.
- 59 Saba What do you mean Ayat?
- 60 Ayat The opposite edge is under the x-axis. So, it is negative. This means $\sin 270 = -1$

Transcript 6: constructing sub-narratives related to the narrative $\sin \alpha = 1$

It seems that the way the teacher intervened in [R56] resulted in a commognitive conflict, where the students applied the previous signification of the trigonometric functions (the right-angled triangle) to the new signification (the unit circle) (R57). Telling the students the right narrative [R56] "I tell you that it is not 1, but -1" and requesting explanation, the teacher probably prompted the students to look for a substantiation for her narrative ' $\sin 270 = -1$ ' instead of constructing their own. Constructing their own narrative, the students would have recalled their earlier substantiated narrative 'The sine is the y-coordinate of the intersection point with the circle', so they would have constructed and

substantiated the narrative $\sin 270 = -1$, using the new signification; i.e. the unit circle.

The students constructed and substantiated the rest of the sub-narratives in a way similar to the one they were engaged with when constructing and substantiating the sub-narratives related to the realizations of $\sin \alpha = 1$. This led them to constructing the narrative $\sin(360k + 90) = 1$.

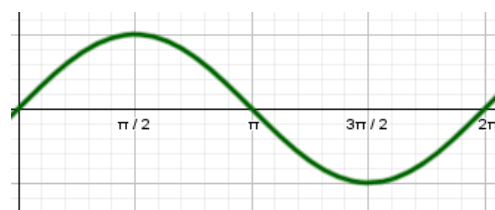
In addition, using the unit circle signification, the students constructed and substantiated the narratives $\sin(360k + 270) = -1$, $\sin \alpha = \sin(180 - \alpha)$, and $\sin(180 + \alpha) = -\sin(\alpha)$. Here, the teacher did not guide the students in any step, where they constructed and substantiated alone the narratives.

Exploring the Period of the Trigonometric Function through Realizing the Coordinate System Signification

The next mathematical activity was to explore the period of a trigonometric function through the coordinate system signification. First, the students used GeoGebra to explore the properties of the graph of the function $\sin \alpha$. Transcript (7) describes students' exploration of the period of $\sin \alpha$.

- 90 Teacher In the previous lesson, we studied the trigonometric functions for angles more than 180 degrees. In the present lesson we will draw the graph of the function $\sin \alpha$ to study the period for which this function goes back on itself.
- 91 Adan Goes back on itself. What does that mean?
- 92 Saba We learned this in the physics class. We learned that the sin and cos functions go back on themselves.
- 93 Adan Right. The teacher told us that these two functions are periodic.
- 94 Ayat What did we see going back on itself? The intersection points with the x-axis, the minimum point and the maximum point.
- 95 Saba This means we need to look when these points go back on themselves.
- 96 Adan Yes, I understand.
- 97 Ayat Right Saba. We need to specify the beginning point.
- 98 Saba We can take zero.
- 99 Ayat Right.

- 100 Adan The graph from zero to two pi has mid intersection point with the x-axis, one maximum point and one minimum point.



- 101 Adan This is also true for the graph in the interval from
- 102 Ayat All the intervals of length two pi, starting from zero, have these three points.
- 103 Saba They asked us about the period of the function.
- 104 Ayat This is the period. The period of $\sin x$ is 2π .
- 105 Adan Yes. I understand.

Transcript 7: students' exploration of the period of $\sin \alpha$

The students used a mix of inductive and deductive processes, as analyzing the properties of the function [94, 100] and comparing between these properties in different intervals of the x-axis [101]. The combination of inductive and deductive processes led the students to deduce the value of the period of the function $\sin x$. This analysis, especially the comparison, helped the students perform sameness of graphs in different intervals of the x-axis [101]. The sameness led to the encapsulation of the properties of the graph [102], and to the reification of the object of period [104].

DISCUSSION

The present research intended to study, using the commognitive framework, grade 10 students' exploration of narratives associated with the sine function in two significations of trigonometric functions; the unit circle and the coordinate system. The research results indicated that the students modified and extended (Sfard, 2007) their right-angled triangle trigonometric discourse through the processes of deduction and induction that were mediated by technology and the teacher's activity. The two processes enabled the students' sameness, encapsulation and reification of trigonometric objects and narratives. Students' inductive processes, through the dragging utility of technology, supported the sameness of signifiers of trigonometric narratives as $\sin 0=0$, $\sin 360=0$ and $\sin 180=0$. Deductive processes, through

comparison, enabled the encapsulation of these signifiers into “the sin of 180 or its multiples equals 0”. Reification of these narratives resulted in “sin 180n = 0, for any integer n”.

The three above factors (students’ routines, teacher’s routines and the software mediation) and their interaction mediated the students’ realizations of the new significations, which helped them realize new trigonometric narratives. The unit circle signification of the trigonometric functions enabled the exploration of angles that are more than 360 degrees, which points at the unit circle signification as appropriate for mediating the students’ rotation concept of the angle (Demir & Heck, 2013). The discursive objects of these angles came into being as a consequence of the teacher’s requests in addition to the students’ utilization of the dragging utility of the applet. Encapsulating and reifying this discursive entity of the angle were performed through the reification of a related object, that of $\sin \alpha$ when the angle is more than 360 degrees. Here, the students’ deductive arguments, as the one about the definition of the sine of an angle in the unit circle, supported their encapsulation and reification processes.

The coordinate system signification of the trigonometric function enabled the exploration of the function’s period concept, where the discursive object of period came into being as a consequence of the combination of inductive and deductive processes. This combination included paying attention to specific properties of the function and comparing between these properties, which led into the awareness of the sameness of parts of the trigonometric function over different intervals, resulting in the encapsulation of the properties of these parts and, as a consequence, the reification of the period as an object.

As described above, deductive and inductive processes helped the students in their realizations of the two new significations. To elaborate more, through performing deductive processes, the students identified right-angled triangle signifiers of trigonometric narratives in their unit-circle realizations. These narratives were general (e.g., the definition of $\sin \alpha$ as the ratio of the length of the opposite side to the length of the hypotenuse) or specific (e.g., “to get $\sin \alpha = 0$, the length of BC should be zero”). In all the deductive processes, technology was a static visual mediator, as it produced the unit circle drawing that mediated the students’ constructing and substantiating of appropriate trigonometric narratives needed for the reification of $\sin \alpha = 0$. These drawings are similar to the ones drawn on paper, but they are easier to produce in the technology environment, especially the measures of angles and the coordinates of the intersection point of the radius with the circumference of the circle.

In addition, technology was a dynamic visual mediator that facilitated students’ engagement in

inductive processes that helped them arrive at numeric realizations of the signifiers of trigonometric objects - as when they identified point five as numeric realization of 30° , or trigonometric narratives - as when they explored specific realizations of the narrative $\sin \alpha = 0$. These inductive processes were mediated by the dragging utility of GeoGebra; specifically through dragging the end point of the hypotenuse; i.e. its intersection point with the unit circle. Here, the mediation was dynamic in the sense that it enabled the production of various angles. Through this production, the students identified numeric values as realizations of signifiers of trigonometric objects or narratives. In addition, through this production, the students, as described above, were able to pay attention the properties of trigonometric functions, and thus could compare between these properties in different intervals.

Furthermore, technology here facilitated the transition of students from inductive reasoning about angles and trigonometric narratives - as when they explored specific realizations of the narrative $\sin \alpha = 0$, to deductive reasoning about these objects and narratives - as the naming of $\sin(540)$ with $\sin(180)$. These findings are in accord with past studies which emphasized the role of the dragging utility in the utilization of the inductive processes for the realization of deductive reasoning. Drijvers, Monaghan, Thomas and Trouche (2015) remind of the ongoing debate about the role of dragging in the move from inductive to deductive reasoning: “there are those who claim this move is often realised and there are those that say dragging is a useful activity to ‘see’ geometric invariants prior to working out a proof without the DGS” (p. 43). The findings of the present research imply that this role of the dragging utility of the software is also substantiated in trigonometric contexts.

The teacher constituted an active factor that impacted positively and sometimes negatively students’ discursive routines. She tried to set norms for the group routines, as the need for the substantiation of narratives in order to endorse them [e.g., explain why $\sin 180 = 0$]. Another routine of the teacher was to complement the text of the mathematical problem, emphasizing what had not been clear in the text, as finding different realizations of the narrative [when the teacher requested the students to see if there are other angles in the unit circle with $\sin \alpha = 0$], or as pointing at routines that the students need to follow, as working with the applet. A third routine of the teacher was to introduce the students to new realizations of a trigonometric narrative when they had not arrived at them alone, as finding $\sin(540)$. The teacher’s routines mediated the exploration activity of the students. In spite of the teacher’s overall positive intervention, her intervention sometimes led to commognitive conflict without advancing the students into understanding the trigonometric objects and narratives in light of the new trigonometric signification.

This happened for example when the teacher corrected the students and told them the right narrative, which led the students to substantiate the teacher's narrative using the old signification. This intervention resulted in the inability of the students to work with the new signification, which made them not able to use mathematical words, and instead, they used everyday words to substantiate the mathematical narratives. The previously described routine of the students and the teacher could be explained also by the claim of Berger (2013) that within the context of technology-enriched mathematical learning, mathematical narratives are positively endorsed only if they agree with the traditional mathematical narratives. Here the students tried to blend between the traditional mathematical signification (the right-angled triangle) and everyday life in order to substantiate the teacher's narrative. In addition, the present research results do not agree with some previous reports that knowing the right answer facilitates the resolution of commognitive conflict (See, for example, Presmeg (2016). Here, knowing the right answer got the students in the commognitive conflict.

CONCLUSIONS

The present research studied, using the commognitive framework, students' routines and mathematizing processes to broaden their conceptions of trigonometric objects through realizing them in different significations. The group members started their exploration of the new significations performing ritual and deed routines that turned into exploration routines. This advancement of the routines from rituals or deeds into explorations is emphasized in Lavie, Steiner and Sfard (2018) who argued that "germinal routines, from which a discourse new to the learner is to emerge, are initially implemented as rituals". Furthermore, they argued that helping students in transforming initial rituals into explorations is among the principal challenges in teaching mathematics. The teacher in the present study generally advanced the transformation of the initially ritual routines into exploration routines, but sometimes she did not succeed to do so, as when she intervened to tell the students the right narrative instead of letting them explore it deductively. In both cases, the teacher's activity took critical role in the exploration activity of the students as an inquiry-mathematics-group (Siegel & Borasi, 1994). These results indicate that the mathematics teacher needs to be aware of the sequence of routines followed by her or his students. The intervention of the mathematics teacher needs to occur mainly through asking questions. The assessment of students' work needs have formative elements, even when it is summative (Broadbent, Panadero & Boud, 2018).

Furthermore, the group members developed their use of words from daily words into mathematical terms.

It is noted that in spite of their daily use of mathematical words, the students endorsed successfully trigonometric narratives using a combination of inductive and deductive processes. These inductive and deductive processes enabled them to perform naming of mathematical processes and properties, which resulted in their encapsulation and reification into mathematical objects, whether words or narratives. The later three processes were mediated by technology. Here too we notice the role of the mathematics teacher who can advance students' learning combining between inductive and deductive processes. Again, the awareness of the mathematics teacher to the positive role of the combination between inductive and deductive processes is needed.

Technology played a mediating role in students' inductive and deductive processes. It played as a dynamic visual mediator for students' inductive processes through its dragging utility. Specifically, the dragging utility mediated the encapsulation of different realizations into a trigonometric narrative. This potential of the dragging utility is emphasized in Ng (2016) who described how dragging mediated a student's encapsulation of a set of ordered pairs into a singular discursive object. In addition, technology played as a static visual mediator for students' deductive processes. The previous results indicate the positive role of technology in students' learning of mathematics, especially in providing dynamic and static visual mediators that support students' exploration of mathematical ideas. The mathematics teacher could depend on technology as mediating her or his students' learning of mathematics.

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