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Optimal control of linear systems with balanced reduced-order models: Perturbation approximations

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ABSTRACT

In this article we study balanced model reduction of linear systems for feedback control problems. Specifically, we focus on linear quadratic regulators with collocated inputs and outputs, and we consider perturbative approximations of the dynamics in the case that the Hankel singular values corresponding to the hardly controllable and observable states go to zero. To this end, we consider different perturbative scenarios that depend on how the negligible states scale with the small Hankel singular values, and derive the corresponding limit systems as well as approximate expressions for the optimal feedback controls. Our approach that is based on a formal asymptotic expansion of an algebraic Riccati equations associated with the Pontryagin maximum principle and that is validated numerically shows that model reduction based on open-loop balancing can also give good closed-loop performance.

1. Introduction

Balanced model reduction, specifically balanced truncation and residualisation, are powerful methods to reduce the dimensionality of large-scale linear open-loop control systems [1]. The idea is to compute an associated pair of Lyapunov equations and identify a subspace that contains only states that are at the same time highly controllable and observable. One of the features is that they give computable, yet relatively conservative *a priori* error bounds for all measurable control inputs with finite energy (i.e. for all square integrable inputs).

It is less clear, however, whether balancing leads to high fidelity reduced models when the inputs are feedback controls that depend on the system states: the reason for scepticism is that model reduction of open-loop systems aims at approximating the system output as a function of the input where in case of partially observable closed-loop systems the input (i.e. the control) is a function of the output. The key question therefore is whether balancing can guarantee a backward stable approximation of the dynamics in the sense of approximating the control.

The linear quadratic regulator (LQR) is a special case of optimal control problem that has an analytic solution in terms of a linear feedback law and a pair of matrix Riccati equations. The design parameters for the LQR are the weighting matrices in the objective function, selected according to the system design. These matrices directly affect the optimal control performance many and discussions in the pas were related to the question how to shape these matrices based on what is called *eigenstructure assignment* [4,6,9]. For finite time-horizon optimal problems, one of the most actively investigated model

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reduction schemes is the singular perturbation approximation, based on a perturbative approximations of the corresponding Riccati differential equation [12]; an alternative approach via two-point boundary value problems is presented in [17] and compared to the former in [18].

Balanced model reduction based on balancing a pair of algebraic control and filter Riccati equations has been first studied by Jonckheere and Silverman [8], based on the idea of projecting the dynamics onto a jointly dominant subspace of the solutions to the two algebraic Riccati equations (ARE); cf. also [19,23]. Compared to standard balanced truncation, LQG balancing is far more expensive as it requires to compute a pair of ARE rather than just a pair of linear Lyapunov equations. Moreover the Riccati equations are lacking the intuitive energy interpretation of the quadratic forms formed by the controllability and observability Gramians that are the solutions to the associated Lyapunov equations.

In this article we follow an alternative approach and ask how to systematically reduce a linear feedback control system to the dominant jointly observable and controllable subspace that is related to the controllability and observability Gramians and the corresponding Hankel singular values of the system. Specifically, we identify the limit LQR system that is obtained from the original dynamics when some of the Hankel singular values go to zero.

The approach pursued in this paper is based on a formal asymptotic expansion of the Pontryagin maximum principle and the associated ARE and gives rise to a reduced-order value function that can be identified with the control value of the balanced reduced-order system. We distinguish three different scenarios that differ in the way that the small Hankel singular values enter the balanced dynamics and which lead to different limit systems. Even though the reduced dynamics is based on open-loop balancing, the reduced systems show good closed-loop performance, and we validate the formal calculations by suitable numerical experiments.

The article is structured as follows: In Section 2 the linear quadratic regulator and the balanced representation of the state space system are introduced. Section 3 that contains the main results is devoted to the formal perturbation analysis of three different classes of singularly perturbed regulator problems, and the fidelity of the resulting reduced closed-loop control systems is compared numerically in Section 4. The findings are briefly summarised in Section 5. The article contains two appendices that recorded various standard results about transfer functions of linear systems and their singular perturbation approximation.

2. Linear quadratic regulator

We consider the continuous linear dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D^{p\times m}$ are constant matrices and x, u are the state and the input of the system and $x(0)$ represents the initial condition. We assume that the linear system described by Eq. (1) is controllable and observable, and we define the quadratic cost function J is defined by

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T R u)dt \tag{2}$$

where $y = Cx$ and $R \in \mathbb{R}^{m\times m}$ is positive definite. We want to find an optimal control u that minimises the quadratic cost function J subject to (1). We seek an optimal control denoted by u^* that has the property that

$$J(u^*) \le J(u), \quad \forall u \in L^2$$

where $u^* \in L^2$ and the constraint equation $\dot{x} = Ax + Bu$ has a unique solution. The corresponding optimal solution of this equation is denoted by x^*.

Now, we introduce an approach that depends on the Hamiltonian function defined in the following form:

$$H = \frac{1}{2}\left(x^T Q x + u^T R u\right) + \lambda^T (Ax + Bu) \tag{3}$$

where $\lambda \in \mathbb{R}^n$ is called the costate variable. The following theorem describes the way in which we can find the optimal control that minimises the quadratic cost J.

Theorem 1. *[10,16] (Maximum principle) If x^*, u^* is an optimal solution of (1) and (2), then there exists a function $\lambda^*(\cdot) \in \mathbb{R}^n$ such that*

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{4}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{5}$$

and the minimality condition of the Hamiltonian

$$H(x^*, u^*, \lambda^*) \le H(x^*, u, \lambda^*)$$

holds for all $u \in \mathbb{R}^m$

For more details on the proof (see [10,16]).

If H is a differentiable function, then to minimise H with respect to u we can find our optimal control input. The following condition must be true to find such u:

$$\frac{\partial H}{\partial u} = 0 \tag{6}$$

If we solve Eq. (6), we obtain the following control:

$$u = -R^{-1}B^T\lambda \tag{7}$$

From (1) and (7), we have the following canonical differential equations that form a linear system (or Hamiltonian system) written as:

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}B^T\lambda, \quad x(0) = x_0 \\ \dot{\lambda} &= -Qx - A^T\lambda \end{aligned} \tag{8}$$

This is a coupled system, linear in x and λ, of order $2n \times 2n$. Since the terminal cost is not defined, then there is no constraint on the final value of λ. The control equations can be written in matrix form as:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{9}$$

The linearity of the equation suggests to use the Ansatz

$$\lambda = Px \tag{10}$$

where $P(\cdot) \in \mathbb{R}^{n \times n}$ is governed by the *differential Riccati equation*

$$\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q \tag{11}$$

Since $u \in L^2$ over an infinite time horizon, it follows that [16].

$$\lim_{t \longrightarrow \infty} \dot{P} = 0$$

which implies that we can replace P in (11) by the unique and positive definite solution to the algebraic Riccati equation (ARE)

$$PA + A^TP - PBR^{-1}B^TP + Q = 0\,. \tag{12}$$

We want now to find a state feedback control u that can be used to move any state x to the origin, so we let the system evolve in a closed-loop [5,16].

If we find the solution P of the ARE (12), then the optimal control u that can be used to minimise the quadratic cost function J is written as:

$$u = -R^{-1}B^TPx \tag{13}$$

By substituting Eq. (13) into the original system described by Eq. (1), we get the following equation:

$$\dot{x} = (A - R^{-1}B^TP)x \tag{14}$$

We can summarise the *LQR* method as follows:

(1) We start with the linear dynamical system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned}$$

(2) We assume that this system is controllable.

(3) We define the quadratic cost function as:

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt$$

(4) We choose $Q = Q^T \ge 0$ such that $Q = C^TC$ and $R = R^T > 0$

(5) We find the constant solution P of the *ARE*:

$$PA + A^TP - PBR^{-1}B^TP + Q = 0$$

(6) We find the optimal control u such that:

$$u = -R^{-1}B^T Px$$

(7) We write the original system into the form:

$$\dot{x} = (A - R^{-1}B^T P)x$$

2.1. Balanced representation

We start from the balanced representation of the linear continuous system to derive a version of Eq. (1) with reduced dimension. The associated controllability and observability Gramians W_c and W_o are positive definite solutions to the pair of Lyapunov equations

$$AW_c + W_c A^T + BB^T = 0$$
$$A^T W_o + W_o A + C^T C = 0$$

In balanced form that can be obtained by a state transformation $x \mapsto Tx$ (see [1]), the two Gramians are equal and diagonal, $W_o = W_c = \Sigma$ where the balanced Gramian Σ can be partitioned in the following form

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

The two partitions

$$\Sigma_1 = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

and

$$\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n)$$

show us the important singular values that we are interested in and the unimportant ones which we want to delete [20,22].

Also we introduce, as in [3], the balance transformation S that satisfies the equations

$$S = UY\Sigma^{\frac{-1}{2}}$$
$$S^{-1} = \Sigma^{\frac{-1}{2}} X^T L^T$$

Now, if we suppose $\sigma_{r+1} << \sigma_r$ and we know that the Hankel singular values (HSVs) are coordinate invariant, then a reduced dimension system with small parameters can be obtained since $\sigma_{r+1} > \sigma_{r+2} > \cdots > \sigma_n > 0$ [7].

To see where the small parameter Σ_2 enter the equation, we replace Σ_2 by $\epsilon\Sigma_2$ or in other words the small HSVs are scaled uniformly according to the equation

$$(\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n) \longmapsto \epsilon(\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n), \quad \epsilon > 0$$

We use the balance transformation $S(\epsilon)$ to change the coordinate such that

$$x \longmapsto S(\epsilon)x$$

If we let $S^{-1}(\epsilon) = T(\epsilon)$, then the balanced matrices are partitioned in the following form [7]:

$$S(\epsilon) = \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \tag{15}$$

and the inverse

$$T(\epsilon) = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \tag{16}$$

If we use the balance transformation described in Eqs. (15) and (16), then a new balance coefficient is obtained and written as:

$$\tilde{A}(\epsilon) = T(\epsilon)AS(\epsilon) = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \frac{1}{\sqrt{\epsilon}}\tilde{A}_{12} \\ \frac{1}{\sqrt{\epsilon}}\tilde{A}_{21} & \frac{1}{\epsilon}\tilde{A}_{22} \end{pmatrix} \tag{17}$$

$$\tilde{B}(\epsilon) = T(\epsilon)B = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \tilde{B}_1 \\ \frac{1}{\sqrt{\epsilon}}\tilde{B}_2 \end{pmatrix} \tag{18}$$

and

$$\tilde{C}(\epsilon) = CS(\epsilon) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}} S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}} S_{22} \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 & \frac{1}{\sqrt{\epsilon}} \tilde{C}_2 \end{pmatrix} \tag{19}$$

If we set $\epsilon = 1$ in Eq. (17), then the value of $\tilde{A} = T(1)AS(1)$ is simply the balanced matrix A. We can rewrite the balancing transformations in the following form

$$S(\epsilon) = S(1)\chi(\epsilon)$$

and

$$T(\epsilon) = \chi(\epsilon)T(1)$$

where

$$\chi(\epsilon) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\epsilon}} \mathbf{I} \end{pmatrix}$$

In the next steps we omit the tilde from the balanced matrices, in order to have the following matrices:

$$A = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}} A_{12} \\ \frac{1}{\sqrt{\epsilon}} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}} B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}} C_2 \end{pmatrix}$$

Let us define the new variable $q = (q_1, q_2)$ which can be balanced using the balance transformation $T(\epsilon)$ and we write q in the balance form as:

$$q = T(\epsilon)x$$

Now, the linear dynamical system in Eq. (1) is converted to the singular perturbation system that is described in the following equation:

$$\begin{aligned} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}} A_{12} \\ \frac{1}{\sqrt{\epsilon}} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}} B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}} C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{20}$$

Eq. (20) can be written in another form:

$$\begin{aligned} \dot{q}_1 &= A_{11} q_1 + \frac{1}{\sqrt{\epsilon}} A_{12} q_2 + B_1 u \\ \dot{q}_2 &= \frac{1}{\sqrt{\epsilon}} A_{21} q_1 + \frac{1}{\epsilon} A_{22} q_2 + \frac{1}{\sqrt{\epsilon}} B_2 u \\ y &= C_1 q_1 + \frac{1}{\sqrt{\epsilon}} C_2 q_2 \end{aligned} \tag{21}$$

the variable q_2 is scaled as

$$q_2 \longmapsto \sqrt{\epsilon} q_2$$

then Eq. (21) becomes:

$$\begin{aligned} \dot{q}_1 &= A_{11} q_1 + A_{12} q_2 + B_1 u \\ \epsilon \dot{q}_2 &= A_{21} q_1 + A_{22} q_2 + B_2 u \\ y &= C_1 q_1 + C_2 q_2 \end{aligned} \tag{22}$$

This system can be written in matrix form as:

$$\begin{aligned} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{23}$$

where the block matrices $A_{11}, A_{12}, \ldots$ are in balance form and ϵ is a small positive scalar that represent all small parameters to be neglected [7,12]. To reduce the dimension of the original system and obtain a reduced order model, we set the singular perturbation $\epsilon = 0$. The linear dynamical system has a multi-time behaviour caused by the singular perturbation and this yields the slow and fast variable of the system. The quasi-steady-state for both slow and fast variables are found with more details in [12]. Now, to apply the singular perturbation approximation and obtain a reduced order model, we introduce the following two assumptions [12]:

Assumptions 2. The block matrix A_{22} is invertible and stable. i.e,

$$\Re\{\lambda(A_{22})\} < 0$$

Assumptions 3. The following equation has a distinct root when we set $\epsilon = 0$.

$$\epsilon\dot{q}_2 = A_{21}q_1 + A_{22}q_2 + B_2u \tag{24}$$

In our dynamical system described by Eq. (22), the slow variable (or dynamic) is q_1 and the fast variable (or dynamic) is q_2. According to the two Assumptions 2, 3 and from Eq. (22), if we set $\epsilon = 0$, then the root of Eq. (24) denoted by $\bar{q}_2$ is given as

$$\bar{q}_2 = -A_{22}^{-1}A_{21}\bar{x} - A_{22}^{-1}B_2u \tag{25}$$

If we substitute the value of $\bar{q}_2$ in the first part of Eq. (22), we obtain the reduced order model represented by the following state-space equation

$$\begin{aligned} \dot{\bar{q}}_1 &= \bar{A}\bar{q}_1 + \bar{B}u \\ \bar{y} &= \bar{C}\bar{q}_1 + \bar{D}u \\ \bar{q}_1(0) &= q_1(0) \end{aligned} \tag{26}$$

where

$$\begin{aligned} \bar{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ \bar{B} &= B_1 - A_{12}A_{22}^{-1}B_2 \\ \bar{C} &= C_1 - C_2A_{22}^{-1}A_{21} \\ \bar{D} &= -C_2A_{22}^{-1}B_2 \end{aligned} \tag{27}$$

3. Singularly perturbed regulator problem

In this section we introduce the linear quadratic regulator problem for the reduced order model of a dynamical system [11]. Our goal is to find an optimal control for the reduced system using the singular perturbation approximation. For the ease of notation we use the original variable names x, y for the original and the balanced systems.

3.1. Singular perturbation regulator problem of type 1

Consider the linear time-invariant dynamical system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix}u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{28}$$

This system can be written in another form as:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1u \\ \epsilon\dot{z} &= A_{21}x + A_{22}z + B_2u \end{aligned} \tag{29}$$

From [3], we see that this system can be optimised according to the following quadratic cost function:

$$J = \frac{1}{2}\int_0^\infty (y^Ty + u^TRu)dt \tag{30}$$

or

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt \tag{31}$$

where $Q = C^TC \geq 0$ and $R > 0$.

The optimal control u is defined as:

$$u = -R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix}P\begin{pmatrix} x \\ z \end{pmatrix} \tag{32}$$

where P is the solution of the Algebraic Riccati equation (ARE):

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{33}$$

The goal now is to solve the *ARE* and set $\epsilon = 0$ to obtain a reduced equation for the *ARE*.

If we substitute the matrices A, B, C and Q in Eq. (33), then we have the following new form of ARE:

$$P\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix}P - P\begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix}R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix}P + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \tag{34}$$

A solution of Eq. (34) can be chosen as:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \tag{35}$$

so we can avoid the unboundedness when we set $\epsilon \longrightarrow 0$ [11].

Substituting Eq. (35) into Eq. (34), we get:

$$\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}$$
$$-\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix}R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \tag{36}$$

From Eq. (36), we get the following $(n+m)\times(n+m)$ equations:

$$0 = P_{11}A_{11} + P_{12}A_{21} + A_{11}^TP_{11} + A_{21}^TP_{12}^T - (P_{11}B_1 + P_{12}B_2)R^{-1}(B_1^TP_{11} + B_2^TP_{12}^T) + C_1^TC_1 \tag{37}$$

$$0 = P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^TP_{12} + A_{21}^TP_{22} - (P_{11}B_1 + P_{12}B_2)R^{-1}(\epsilon B_1^TP_{12} + B_2^TP_{22}) + C_1^TC_2 \tag{38}$$

$$0 = \epsilon P_{12}^TA_{11} + P_{22}A_{21} + A_{12}^TP_{11} + A_{22}^TP_{12}^T - (\epsilon P_{12}^TB_1 + P_{22}B_2)R^{-1}(B_1^TP_{11} + B_2^TP_{12}^T) + C_2^TC_1 \tag{39}$$

$$0 = \epsilon P_{12}^TA_{12} + P_{22}A_{22} + \epsilon A_{12}^TP_{12} + A_{22}^TP_{22} - (\epsilon P_{12}^TB_1 + P_{22}B_2)R^{-1}(\epsilon B_1^TP_{12} + B_2^TP_{22}) + C_2^TC_2 \tag{40}$$

When we set $\epsilon = 0$ in Eqs. (37)–(40) we obtain the following $m\times m$ reduced equation for $\bar{P}_{22}$ and written as:

$$\bar{P}_{22}A_{22} + A_{22}^T\bar{P}_{22} - \bar{P}_{22}W\bar{P}_{22} + C_2^TC_2 = 0 \tag{41}$$

where $W = B_2R^{-1}B_2^T$.

Another $n\times n$ equation for $\bar{P}_{11}$ is obtained when we express $\bar{P}_{12}$ in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ and this equation takes the form:

$$\bar{P}_{11}\hat{A} + \hat{A}^T\bar{P}_{11}^T - \bar{P}_{11}\hat{B}R^{-1}\hat{B}^T\bar{P}_{11} + \hat{C}^T\hat{C} = 0 \tag{42}$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are defined in [14]. If $(\hat{A},\hat{B})$ is controllable pair and $(\hat{A},\hat{C})$ is observable pair, then applying the implicit function theorem to Eq. (34) with Eq. (35) [11,12], we have:

$$P_{ij} = \bar{P}_{ij} + O(\epsilon), \qquad i,j = 1,2 \tag{43}$$

If we use $\bar{P}_{ij}$ instead of P_{ij} in Eq. (43), then the feedback control in Eq. (32) becomes:

$$\begin{aligned} u &= -R^{-1}\begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix}\begin{pmatrix} \bar{P}_{11} & \epsilon\bar{P}_{12} \\ \epsilon\bar{P}_{12}^T & \epsilon\bar{P}_{22} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1}(B_1^T\bar{P}_{11} + B_2^T\bar{P}_{12})x - R^{-1}(\epsilon B_1^T\bar{P}_{12} + B_2^T\bar{P}_{22})z \end{aligned} \tag{44}$$

From Eq. (44), the original system described by Eq. (28) becomes:

$$\begin{aligned} \dot{x} &= \left(A_{11} - B_1R^{-1}(B_1^T\bar{P}_{11} + B_2^T\bar{P}_{12})\right)x + \left(A_{12} - B_1R^{-1}(\epsilon B_1^T\bar{P}_{12} + B_2^T\bar{P}_{22})\right)z \\ \epsilon\dot{z} &= \left(A_{21} - B_2R^{-1}(B_1^T\bar{P}_{11} + B_2^T\bar{P}_{12})\right)x + \left(A_{22} - B_2R^{-1}(\epsilon B_1^T\bar{P}_{12} + B_2^T\bar{P}_{22})\right)z \end{aligned} \tag{45}$$

If this system is asymptotically stable then from Eq. (43), we have a solution $x(t)$ and $z(t)$ with $O(\epsilon)$ of the optimal solution [13]. If we assume that A_{22} is stable, then we can apply this assumption to the feedback system in Eq. (45).

If we reduce the full system in Eq. (29) using the singular approximation approximation, we obtain the following reduced order model:

$$\begin{aligned} \dot{x}_r &= A_rx_r + B_ru_r \\ y_r &= C_rx_r + D_ru_r \end{aligned} \tag{46}$$

where

$$A_r = A_{11} - A_{12}A_{22}^{-1}A_{21}$$
$$B_r = B_1 - A_{12}A_{22}^{-1}B_2$$

$$C_r = C_1 - C_2 A_{22}^{-1} A_{21}$$
$$D_r = -C_2 A_{22}^{-1} B_2$$

We define the cost quadratic function of this reduced order system as:

$$J_r = \frac{1}{2} \int_0^\infty (y_r^T y_r + u_r^T R_r u_r) dt \quad (47)$$

or, equivalently

$$J_r = \frac{1}{2} \int_0^\infty (x_r^T Q_r x_r + 2 x_r^T C_r D_r u_r + u_r^T R_r u_r) dt \quad (48)$$

where $Q_r = C_r^T C_r$ and $R_r = R + D_r^T D_r$.

The optimal control for this reduced system defined as:

$$u_r = -R_r^{-1} B_r^T P_r x_r \quad (49)$$

where P_r is the constant solution of the following *Algebraic Riccati equation* for the reduced system described by Eq. (46) given as:

$$P_r (A_r - B_r R_r^{-1} D_r^T C_r) + (A_r - B_r R_r^{-1} D_r^T C_r)^T P_r - P_r B_r R_r^{-1} B_r^T P_r + C_r^T (I + D_r R_r D_r^T)^{-1} C_r = 0 \quad (50)$$

We introduce now the following theorem that describes the relationship between the reduced Riccati equation system (41) and (42) for the full system (29) after putting $\epsilon = 0$ and the Riccati equation (50) for the reduced system in Eq. (46) when we set $\epsilon = 0$

Theorem 4. *If Eq. (43) holds and A_{22}^{-1} exists, then the solution P_r of Eq. (50) is identical to the solution $\bar{P}_{11}$ of Eq. (42).*

For more details, see [12,13].

According to Theorem 4 and if we substitute the feedback optimal control u_r described by Eq. (49) into the reduced system Eq. (46), then we obtain the following system:

$$\dot{x}_r = (A_r - B_r R^{-1} B_r^T P_r) x_r \quad (51)$$

where $(A_r - B_r R^{-1} B_r^T P_r)$ is stable and the pair (A_r, B_r) is controllable. If we find the optimal solution x_r of (51) and substitute the value into Eq. (49), then we find the optimal control for the reduced order model.

3.2. Singular perturbation regulator problem of type 2

In this subsection, we introduce a linear dynamical continuous system with input matrix B that does not depends on ϵ. We want to find the optimal control for this dynamical system and then use the singular perturbation approximation to reduce this system and find the optimal control for the reduced order model.

Let us consider the following linear dynamical continuous system defined as:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad (52)$$

Another representation of the above system could be written as:

$$\begin{aligned} \dot{x} &= A_{11} x + A_{12} z + B_1 u \\ \epsilon \dot{z} &= A_{21} x + A_{22} z + \epsilon B_2 u \end{aligned} \quad (53)$$

If we assume that A_{22} is stable and A_{22}^{-1} exists, then we set $\epsilon = 0$ to obtain the following equation:

$$\bar{z} = A_{22}^{-1} A_{21} \bar{x} \quad (54)$$

When we substitute Eq. (54) into Eq. (53), we get the following reduced order model:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A} \bar{x} + \bar{B} \bar{u} \\ \bar{y} &= \bar{C} \bar{x} \end{aligned} \quad (55)$$

where

$$\bar{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}$$
$$\bar{B} = B_1$$
$$\bar{C} = C_1 - C_2 A_{22}^{-1} A_{21}$$

Our goal now is to find the optimal control for the system in Eq. (52) that minimises the quadratic cost function J defined by the following equations:

$$J=\frac{1}{2}\int_0^\infty (y^Ty+u^TRu)dt \tag{56}$$

or equivalently

$$J=\frac{1}{2}\int_0^\infty (x^TQx+u^TRu)dt \tag{57}$$

where $Q=C^TC\geq 0$ and $R>0$.

The feedback optimal control u for the original system is defined as:

$$u=-R^{-1}\begin{pmatrix}B_1^T & B_2^T\end{pmatrix}P\begin{pmatrix}x\\ z\end{pmatrix} \tag{58}$$

where P is the solution of the Algebraic Differential Equation defined below:

$$P\begin{pmatrix}A_{11} & A_{12}\\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22}\end{pmatrix}+\begin{pmatrix}A_{11}^T & \frac{1}{\epsilon}A_{21}^T\\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T\end{pmatrix}P-P\begin{pmatrix}B_1\\ B_2\end{pmatrix}R^{-1}\begin{pmatrix}B_1^T & B_2^T\end{pmatrix}P+\begin{pmatrix}C_1^T\\ C_2^T\end{pmatrix}\begin{pmatrix}C_1 & C_2\end{pmatrix}=0 \tag{59}$$

We choose the solution of Eq. (59) as:

$$P=\begin{pmatrix}P_{11} & \epsilon P_{12}\\ \epsilon P_{12}^T & \epsilon P_{22}\end{pmatrix} \tag{60}$$

to avoid the unboundedness for $\epsilon=0$.

Eq. (60) together with Eq. (59) give the following equation:

$$\begin{aligned}&\begin{pmatrix}P_{11} & \epsilon P_{12}\\ \epsilon P_{12}^T & \epsilon P_{22}\end{pmatrix}\begin{pmatrix}A_{11} & A_{12}\\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22}\end{pmatrix}+\begin{pmatrix}A_{11}^T & \frac{1}{\epsilon}A_{21}^T\\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T\end{pmatrix}\begin{pmatrix}P_{11} & \epsilon P_{12}\\ \epsilon P_{12}^T & \epsilon P_{22}\end{pmatrix}\\ &-\begin{pmatrix}P_{11} & \epsilon P_{12}\\ \epsilon P_{12}^T & \epsilon P_{22}\end{pmatrix}\begin{pmatrix}B_1\\ \frac{1}{\epsilon}B_2\end{pmatrix}R^{-1}\begin{pmatrix}B_1^T & \frac{1}{\epsilon}B_2^T\end{pmatrix}\begin{pmatrix}P_{11} & \epsilon P_{12}\\ \epsilon P_{12}^T & \epsilon P_{22}\end{pmatrix}+\begin{pmatrix}C_1^T\\ C_2^T\end{pmatrix}\begin{pmatrix}C_1 & C_2\end{pmatrix}=0\end{aligned} \tag{61}$$

Form Eq. (61), we obtain the following set of equations:

$$0=P_{11}A_{11}+P_{12}A_{21}+A_{11}^TP_{11}+A_{21}^TP_{12}^T-(P_{11}B_1+\epsilon P_{12}B_2)R^{-1}(B_1^TP_{11}+\epsilon B_2^TP_{12}^T)+C_1^TC_1 \tag{62}$$

$$0=P_{11}A_{12}+P_{12}A_{22}+\epsilon A_{11}^TP_{12}+A_{21}^TP_{22}-(P_{11}B_1+\epsilon P_{12}B_2)R^{-1}(\epsilon B_1^TP_{12}+\epsilon B_2^TP_{22})+C_1^TC_2 \tag{63}$$

$$0=\epsilon P_{12}^TA_{11}+P_{22}A_{21}+A_{12}^TP_{11}+A_{22}^TP_{12}^T-(\epsilon P_{12}^TB_1+\epsilon P_{22}B_2)R^{-1}(\epsilon B_1^TP_{11}+\epsilon B_2^TP_{12}^T)+C_2^TC_1 \tag{64}$$

$$0=\epsilon P_{12}^TA_{12}+P_{22}A_{22}+\epsilon A_{12}^TP_{12}+A_{22}^TP_{22}-(\epsilon P_{12}^TB_1+\epsilon P_{22}B_2)R^{-1}(\epsilon B_1^TP_{12}+\epsilon B_2^TP_{22})+C_2^TC_2 \tag{65}$$

When we set $\epsilon=0$ in Eqs. (62)–(65) we obtain the following reduced Riccati equations:

$$\bar{P}_{11}A_{11}+\bar{P}_{12}A_{21}+A_{11}^T\bar{P}_{11}^T+A_{21}^T\bar{P}_{12}^T-\bar{P}_{11}B_1R^{-1}B_1^T\bar{P}_{11}+C_1^TC_1=0 \tag{66}$$

$$\bar{P}_{11}A_{12}+\bar{P}_{12}A_{22}+A_{21}^T\bar{P}_{22}+C_1^TC_2=0 \tag{67}$$

$$\bar{P}_{22}A_{21}+A_{12}^T\bar{P}_{11}+A_{22}^T\bar{P}_{12}^T+C_2^TC_1=0 \tag{68}$$

$$\bar{P}_{22}A_{22}+A_{22}^T\bar{P}_{22}+C_2^TC_2=0 \tag{69}$$

We write $\bar{P}_{12}$ and $\bar{P}_{12}^T$ in Eqs. (67) and (68) in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ as follows:

$$\bar{P}_{12}=-(\bar{P}_{11}A_{12}+A_{21}^T\bar{P}_{22}+C_1^TC_2)A_{22}^{-1} \tag{70}$$

$$\bar{P}_{12}^T=-(A_{22}^T)^{-1}(\bar{P}_{22}A_{21}+A_{12}^T\bar{P}_{11}+C_2^TC_1) \tag{71}$$

Eq. (69) can be expressed in different form as:

$$A_{21}^T(A_{22}^T)^{-1}\bar{P}_{22}A_{21}+A_{21}^T\bar{P}_{22}A_{22}^{-1}A_{21}=-A_{21}^T(A_{22}^T)^{-1}C_2^TC_2A_{22}^{-1}A_{21} \tag{72}$$

Substituting Eqs. (70) and (71) into Eq. (66) and using Eq. (72) we obtain:

$$\bar{P}_{11}\hat{A}+\hat{A}^T\bar{P}_{11}-\bar{P}_{11}\hat{B}R^{-1}\hat{B}^T\bar{P}_{11}+\hat{C}^T\hat{C}=0 \tag{73}$$

where

$$\begin{aligned}\hat{A}&=A_{11}-A_{12}A_{22}^{-1}A_{21}\\ \hat{B}&=B_1\\ \hat{C}&=C_1-C_2A_{22}^{-1}A_{21}\end{aligned} \tag{74}$$

If we assume the pair $(\hat{A},\hat{B})$ is controllable, then the values of P_{ij} and $\bar{P}_{ij},\quad i,j=1,2$ satisfy Eq. (43).

The feedback optimal control defined in Eq. (58) together with the result in Eq. (43) can be written as:

$$\begin{aligned}u&=-R^{-1}\begin{pmatrix}B_1^T & B_2^T\end{pmatrix}\begin{pmatrix}\bar{P}_{11} & \epsilon\bar{P}_{12}\\ \epsilon\bar{P}_{12}^T & \epsilon\bar{P}_{22}\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix}\\ &=-R^{-1}(B_1^T\bar{P}_{11}+\epsilon B_2^T\bar{P}_{12})x-R^{-1}(\epsilon B_1^T\bar{P}_{12}+\epsilon B_2^T\bar{P}_{22})z\end{aligned} \tag{75}$$

We can use the result found in Eq. (75) to write a new representation of the original system described by Eq. (53) as:

$$\begin{aligned}\dot{x}&=\big(A_{11}-B_1R^{-1}(B_1^T\bar{P}_{11}+\epsilon B_2^T\bar{P}_{12})\big)x+\big(A_{12}-B_1R^{-1}(\epsilon B_1^T\bar{P}_{12}+\epsilon B_2^T\bar{P}_{22})\big)z\\ \epsilon\dot{z}&=\big(A_{21}-\epsilon B_2R^{-1}(B_1^T\bar{P}_{11}+\epsilon B_2^T\bar{P}_{12})\big)x+\big(A_{22}-\epsilon B_2R^{-1}(\epsilon B_1^T\bar{P}_{12}+\epsilon B_2^T\bar{P}_{22}\big)z\end{aligned} \tag{76}$$

If the system in Eq. (76) is asymptotically stable and if Eq. (43) holds, then we can compute the solution $x(t)$ and $z(t)$ within the $O(\epsilon)$ of the optimal control.

The next step now is to find a feedback optimal control for the reduced system defined in Eq. (55) that can be used to minimises the quadratic cost function $\bar{J}$ defined as:

$$\bar{J}=\frac{1}{2}\int_0^\infty(\bar{y}^T\bar{y}+\bar{u}T\bar{R}\bar{u})dt \tag{77}$$

or equivalently

$$\bar{J}=\frac{1}{2}\int_0^\infty(\bar{x}^T\bar{Q}\bar{x}+\bar{u}^T\bar{R}\bar{u})dt \tag{78}$$

where $\bar{Q}=\bar{C}^T\bar{Q}\geq 0$ and $\bar{R}=R>0$.

We define the optimal control for the reduced system (55) as:

$$\bar{u}=-\bar{R}^{-1}\bar{B}^T\bar{P}\bar{x} \tag{79}$$

where $\bar{P}$ is the solution of the following Algebraic Riccati equation for the reduced system in Eq. (55), defined as:

$$\bar{P}\bar{A}+\bar{A}^T\bar{P}-\bar{P}\bar{B}\bar{R}^{-1}\bar{B}^T\bar{P}+\bar{C}^T\bar{C}=0 \tag{80}$$

Since A_{22} is stable and A_{22}^{-1} is exists, then the solution of Eq. (80) is the same as the solution of Eq. (73), thus we have:

$$\bar{P}=\bar{P}_{11} \tag{81}$$

By using the feedback optimal control in Eq. (79) and the solution $\bar{P}$ in Eq. (80), then we obtain the following reduced system derived from the reduced system in Eq. (55) that has the form:

$$\begin{aligned}\dot{\bar{x}}&=\big(\bar{A}-\bar{B}\bar{R}^{-1}\bar{B}^T\bar{P}\big)\bar{x}\\ \bar{y}&=\bar{C}\bar{x}\end{aligned} \tag{82}$$

where

$$\begin{aligned}\bar{A}&=A_{11}-A_{12}A_{22}^{-1}A_{21}\\ \bar{B}&=B_1\\ \bar{C}&=C_1-C_2A_{22}^{-1}A_{21}\end{aligned}$$

We assume that the matrix $\bar{A}-\bar{B}\bar{R}^{-1}\bar{B}^T\bar{P}$ is stable and the pairs $(\bar{A},\bar{B})$, $(\bar{A},\bar{C})$ are controllable and observable, respectively.

By solving the reduced system in Eq. (82), the solution $\bar{x}(t)$ is used to find the feedback control $\bar{u}$ which is important to find the minimum value of the quadratic cost function $\bar{J}$.

3.3. Singular perturbation regulator problem of type 3

In Section 3.2, we applied the singular perturbation linear quadratic regulator to find an optimal control for the reduced system. We now turn to a different scaling that shares some features with the classical balanced truncation approach. To this end consider the full linear time-invariant dynamical system defined by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{83}$$

We can rewrite the original system in Eq. (83) in another form as:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ \epsilon \dot{z} &= \epsilon A_{21}x + A_{22}z + \epsilon B_2 u \end{aligned} \tag{84}$$

If we apply the balanced truncation method to reduce the original system described by Eq. (84), we get the following reduced system form:

$$\begin{aligned} \dot{x}_r &= A_{11}x_r + B_1 u_r \\ y_r &= C_1 x_r \end{aligned} \tag{85}$$

Moreover, we can apply the singular perturbation approximation method to reduce the original system in Eq. (84) to obtain the reduced system:

$$\begin{aligned} \dot{\bar{x}} &= A_{11}\bar{x} + B_1 \bar{u} \\ \bar{y} &= C_1 \bar{x} \end{aligned} \tag{86}$$

From Eqs. (85) and (86), we see the two reduced systems have the same state space equation and this means that to find an optimal control for the reduced system in Eq. (85) using the balanced truncation method, we can use the singular perturbation method described in Section 3.2.

We start by defining the quadratic cost function J for the original system (83) as:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T R u)dt \tag{87}$$

or equivalently

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt \tag{88}$$

where $Q = C^T C \geq 0$ and $R > 0$.

Our optimal control u for the original system is defined as:

$$u = -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{89}$$

The matrix P is the solution of the following ARE:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{90}$$

The next step now is to find a reduced Riccati equation for the full Riccati equation (90) when $\epsilon = 0$.

To avoid the unboundedness when $\epsilon = 0$, we choose the solution P in the form:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \tag{91}$$

By substituting Eq. (91) into Eq. (90), we get:

$$\begin{aligned} &\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix} \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &\quad - \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{92}$$

After solving Eq. (92), we obtain the following equations:

$$0 = P_{11}A_{11} + \epsilon P_{12}A_{21} + A_{11}^T P_{11} + \epsilon A_{21}^T P_{12}^T - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(B_1^T P_{11} + \epsilon B_2^T P_{12}^T) + C_1^T C_1 \tag{93}$$

$$0 = P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^T P_{12} + \epsilon A_{21}^T P_{22} - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) + C_1^T C_2 \tag{94}$$

$$0 = \epsilon P_{12}^T A_{11} + \epsilon P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}(B_1^T P_{11} + \epsilon B_2^T P_{12}^T) + C_2^T C_1 \tag{95}$$

$$0 = \epsilon P_{12}^T A_{12} + P_{22}A_{22} + \epsilon A_{12}^T P_{12} + A_{22}^T P_{22} - (\epsilon P_{12}^T B_1 + \epsilon P_{22} B_2) R^{-1} (\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) + C_2^T C_2 \tag{96}$$

Now, if we set $\epsilon = 0$ in Eqs. (93)–(96), we obtain the following reduced system Riccati equations:

$$\bar{P}_{11} A_{11} + A_{11}^T \bar{P}_{11}^T - \bar{P}_{11} B_1 R^{-1} B_1^T \bar{P}_{11} + C_1^T C_1 = 0 \tag{97}$$

$$\bar{P}_{11} A_{12} + \bar{P}_{12} A_{22} + C_1^T C_2 = 0 \tag{98}$$

$$A_{21}^T \bar{P}_{11} + A_{22}^T \bar{P}_{12} + C_2^T C_1 = 0 \tag{99}$$

$$\bar{P}_{22} A_{22} + A_{22}^T \bar{P}_{22} + C_2^T C_2 = 0 \tag{100}$$

Assumptions 5. The pair (A_{11}, B_1) is controllable and $\bar{P}_{11}$ is a unique positive semidefinite solution of Eq. (97) such that:

$$A_{11} - B_1 R^{-1} B_1^T \bar{P}_{11}$$

is stable.

According to Eq. (43) in Section 3.1, we can use $\bar{P}_{ij}$ instead of P_{ij} to rewrite the feedback control in Eq. (89) as:

$$\begin{aligned} u &= -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} \bar{P}_{11} & \epsilon \bar{P}_{12} \\ \epsilon \bar{P}_{12}^T & \epsilon \bar{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1} (B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12}) x - R^{-1} (\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22}) z \end{aligned} \tag{101}$$

Using Eq. (101), we obtain a new form of the original system described by Eq. (84) such that:

$$\begin{aligned} \dot{x} &= \big(A_{11} - B_1 R^{-1} (B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12})\big) x + \big(A_{12} - B_1 R^{-1} (\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22})\big) z \\ \epsilon \dot{z} &= \big(\epsilon A_{21} - \epsilon B_2 R^{-1} (B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12})\big) x + \big(A_{22} - \epsilon B_2 R^{-1} (\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22})\big) z \end{aligned} \tag{102}$$

If the above system is asymptotically stable and Eq. (43) holds, then we have a solution $x(t)$ and $z(t)$ for this system with $O(\epsilon)$ of the optimal solution [13].

We are going now to define the quadratic cost function for the reduced order model system described in Eq. (85) or (86). Let $\bar{J}$ be the quadratic cost function of the reduced system in Eq. (85) or (86) defined as:

$$\bar{J} = \frac{1}{2} \int_0^\infty (\bar{y}^T \bar{y} + \bar{u}^T \bar{R} \bar{u}) dt \tag{103}$$

or, equivalently

$$\bar{J} = \frac{1}{2} \int_0^\infty (\bar{x}^T \bar{Q} \bar{x} + \bar{u}^T \bar{R} \bar{u}) dt \tag{104}$$

where $\bar{Q} = \bar{C}^T \bar{C} \geq 0$ and $\bar{R} = R > 0$.

The optimal feedback control for the reduced order model is defined as:

$$\bar{u} = -\bar{R}^{-1} \bar{B}^T \bar{P} \bar{x} \tag{105}$$

where $\bar{P}$ is the solution of the Algebraic Riccati equation for the reduced order model and given as:

$$\bar{P} A_1 + A_1^T \bar{P} - \bar{P} B_1 \bar{R}^{-1} B_1^T \bar{P} + C_1^T C_1 = 0 \tag{106}$$

From Theorem 4 in Section 3.1, we see that the two solutions $\bar{P}_{11}$ and $\bar{P}$ are both identical.

Hence we conclude that $\bar{P}_{11}$ is the reduced Riccati equation (97) and it is the same as $\bar{P}$ which is the solution of the reduced system.

By substituting the feedback control Eq. (105) into the reduced system (85), we get:

$$\dot{\bar{x}} = (A_{11} - B_1 R^{-1} B_1^T \bar{P}) x_r \tag{107}$$

where we have assumed that the matrix $(A_{11} - \bar{B} R^{-1} B_1^T \bar{P})$ is stable.

If we solve Eq. (107) of the reduced system, then we can use the solution $x(t)$ to find the optimal control. This optimal control can be used to find the optimality of $\bar{J}$.

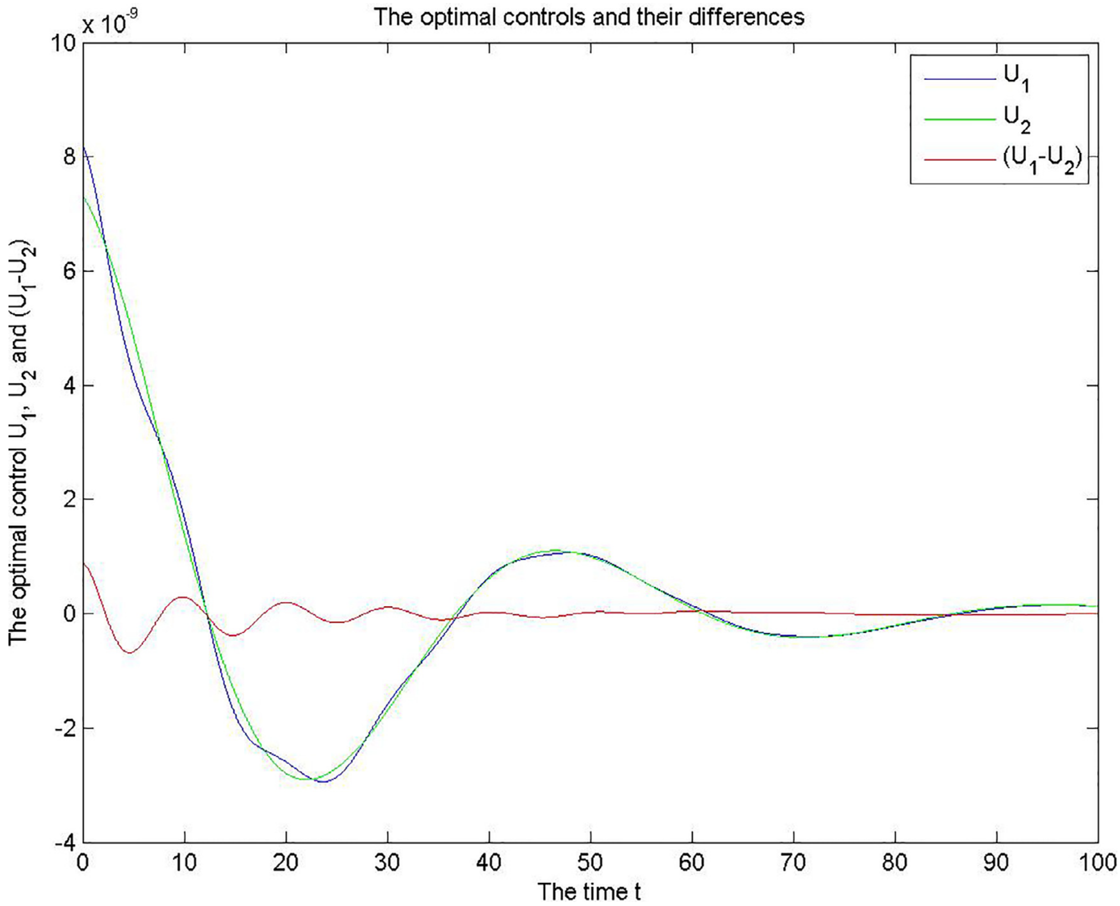

Fig. 1. The optimal controls of the mass-spring damping.

Table 1
The L^2 norm of $(U_1 - U_2)$ and $(P_{11} - \tilde{P}_{11})$ of the mass-spring.

r_s	$\|U_1 - U_2\|_{L_2}$ BT	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$	$\|U_1 - U_2\|_{L_2}$ SPA	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$
2	4.0511×10^{-21}	0.0159	4.2643×10^{-21}	0.0193
4	3.1877×10^{-23}	0.0049	3.4627×10^{-12}	0.0035
6	2.7572×10^{-25}	6.4136×10^{-4}	3.1813×10^{-25}	2.1445×10^{-4}
8	2.2720×10^{-28}	6.9710×10^{-4}	7.0385×10^{-27}	3.8847×10^{-5}
10	4.0902×10^{-28}	1.1015×10^{-4}	1.1291×10^{-29}	6.8224×10^{-6}

4. Numerical illustration

In this section we include all results obtained by the singular perturbation approximation (SPA) techniques to determine the order of the reduced models.

To find an optimal control U_1 for the original system and U_r for the reduced order system, we apply the balanced truncation and the singular perturbation approximation methods to the mass damping system example. The size of the system is $N_s = 10$ and the size of the reduced model is $r_s = 2$. The optimal control is computed by using the results in Sections 3.2 and 3.3. The solution of the Riccati equation P of the full system is computed and used to find the value of U_1. We apply the approaches in Sections 3.2 and 3.3 to find the solution of the Riccati equation P_r of the reduced system. Since the first block P_{11} of P is equal to the value of P_r, so we can extended P using P_r as the first block and the rest blocks are zero to obtain a new solution of the Riccati equation denoted by $\tilde{P}_{11}$.

Another optimal control for the full system U_2 is found using the value of $\tilde{P}_{11}$, and hence we compute the $\|U_1 - U_2\|_{L_2}$ norm. Fig. 1 represent the plots of the two optimal controls U_1, U_2 and $(U_1 - U_2)$ using the balanced truncation and the singular approximation perturbation.

Finally, Table 1 contains the values of $\|U_1 - U_2\|_{L_2}$ and $\|P_{11} - \tilde{P}_{11}\|_{L_2}$ by applying the balanced truncation and singular perturbation approximation to the mass-spring damping.

5. Conclusions

In this article we have discussed singular perturbation approximations of LQR systems based on open-loop balancing of controllability and observability properties. We have studied the behaviour of the control value and the corresponding optimal control in the limit of vanishing Hankel singular values and analysed three scenarios that differ in how the dynamics scales with the negligible Hankel singular values. As two special cases of the singular perturbation approach, we recover the balanced truncation approximation (type 3) as well as the residualisation method (type 1). Even though our approach remained purely formal, we have given some numerical evidence that open-loop balancing can give good closed-loop performance.

Appendix A. The reciprocal system of a linear dynamical system

In this section we discuss some properties and some results related to the reciprocal system of the balanced realisation for the infinite dimensional systems [21].

Let the linear continuous dynamical system represented by the equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

If the system (A, B, C, D) is balanced with Gramian Σ, then we have

$$A\Sigma + \Sigma A^T + BB^T = 0$$
$$A^T\Sigma + \Sigma A + C^TC = 0$$

We let $G(s)$ to be the transfer function of the balanced system (A, B, C, D), then

$$G(s) = C(sI - A)^{-1}B + D$$

the reciprocal system of the balanced system (A, B, C, D) is denoted by $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and defined as [2,21,22]:

$$\begin{aligned} \hat{A} &= A^{-1} \\ \hat{B} &= A^{-1}B \\ \hat{C} &= -CA^{-1} \\ \hat{D} &= D - CA^{-1}B \end{aligned} \tag{108}$$

Remark A.1. If we compute the value of $G(0)$, we have that:

$$G(0) = -CA^{-1}B + D = \hat{D}$$

Remark A.2. Let a matrix A is given as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

the inverse of A is:

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}$$

we also have

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{12}A_{11}^{-1} \\ -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}$$

Let $\hat{G}$ be the transfer function of the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, then:

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{109}$$

the relation between the two transfer functions G and $\hat{G}$ is given as:

$$\begin{aligned} G(S) &= C(sI - A)^{-1}B + D \\ &= C(sI - A)^{-1}AA^{-1}B + D \\ &= C\frac{I}{s}(A^{-1} - \frac{I}{s})^{-1}A^{-1}B + D \\ &= -C\left(\frac{I}{s} - A^{-1} + A^{-1}\right)\left(\frac{I}{s} - A^{-1}\right)^{-1}A^{-1}B + D \\ &= -CA^{-1}B - CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1}A^{-1}B + D \\ &= -CA^{-1}\left(\frac{I}{s} - A^{-1}\right)^{-1}A^{-1}B + D - CA^{-1}B \\ &= \hat{C}\left(\frac{I}{s} - \hat{A}\right)^{-1}\hat{B} + \hat{D} \\ &= \hat{G}\left(\frac{1}{s}\right) \end{aligned} \tag{110}$$

The following lemma shows us the balanced realisation of the reciprocal system [15,21].

Lemma A.3. *Let the system (A, B, C, D) be the minimal and balanced realisation with Gramian Σ of a linear, time-invariant and stable system, then the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is also balanced with the same Gramian Σ.*

Proof. We know that Σ satisfies the Lyapunov equations

$$A\Sigma + \Sigma A^T + BB^T = 0$$
$$A^T\Sigma + \Sigma A + C^TC = 0$$

Thus multiplying the first equation from the right by A^{-1} and from the left by A^{-T} we get

$$\begin{aligned} A^{-1}(A\Sigma)A^{-T} + A^{-1}(\Sigma A^T)A^{-T} + A^{-1}(BB^T)A^{-T} = 0 \\ \Sigma A^{-T} + A^{-1}\Sigma + (A^{-1}B)(A^{-1}B)^T = 0 \end{aligned}$$

Substituting the values in Eq. (108), we have that

$$\hat{A}\Sigma + \Sigma\hat{A}^T + \hat{B}\hat{B}^T = 0$$

The second Lyapunov equation multiplied by A^{-T} from the right and by A^{-1} from the left, gives us

$$\begin{aligned} A^{-T}(A^T\Sigma)A^{-1} + A^{-T}(\Sigma A)A^{-1} + A^{-T}(C^TC)A^{-1} = 0 \\ \Sigma A^{-1} + A^{-T}\Sigma + (CA^{-1})^T(CA^{-1}) = 0 \end{aligned}$$

In the same way from Eq. (108), we have

$$\hat{A}^T\Sigma + \Sigma\hat{A} + \hat{C}^T\hat{C} = 0$$

This means that the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is balanced with the same Gramian Σ. □

The reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and the Gramian Σ are partitioned as

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

then if we refer to [1,15,24], we have the following lemma:

Lemma A.4. *Let the hypothesis of Lemma A.3 hold and let the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ be partitioned as in Eq. (111). Then the subsystems $(\hat{A}_{ii}, \hat{B}_i, \hat{C}_i, \hat{D})$, $\quad i = 1, 2$ are also internally balanced with Gramian Σ_i, $\quad$ for $i = 1, 2$.*

Lemma A.5. *Let the hypothesis of Lemma A.4 hold. Then the subsystem matrices $\hat{A}_{ii}$, $\quad i = 1, 2$ are asymptotically stable if Σ_1 and Σ_2 have no common diagonal element. Further, the subsystem $(\hat{A}_{ii}, \hat{B}_i, \hat{C}_i, \hat{D})$, $\quad i = 1, 2$ is controllable and observable.*

In order to apply balance truncation to the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ we assume that the Hankel singular values σ_j for $j = 1, 2, \ldots, r$ are distinct and such that $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$ to have $\Sigma_1 > 0$, Then we have the following $r \times r$

reduced system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ with state space equation:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}_{11}\hat{x} + \hat{B}_1 u \\ \hat{y} &= \hat{C}_1\hat{x} + \hat{D}u \end{aligned} \tag{111}$$

The values of $\hat{A}_{11}, \quad \hat{B}_1, \quad \hat{C}_1$ and $\hat{D}$ can be computed from Eq. (108) and Remark A.2, and they defined as:

$$\begin{aligned} \hat{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\ \hat{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) \\ \hat{C}_1 &= (C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\ \hat{D} &= D - CA^{-1}B \end{aligned} \tag{112}$$

The transfer function for the reduced system (113) is denoted by $\hat{G}_r$ and defined as:

$$\hat{G}_r(s) = \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1 + \hat{D} \tag{113}$$

We want, now, to find the H_∞ norm for the reduced reciprocal system.

The error bound is represented in the following lemma:

Lemma A.6. *We have*

$$\|\hat{G} - \hat{G}_r\|_\infty \le 2\sum_{i=r+1}^{n}\sigma_i \tag{114}$$

The proof of this lemma can be found in [15].

Appendix B. Singular perturbation approximation

Let $\bar{G}$ be the transfer function of the reduced order model in Eq. (26), then:

$$\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \tag{115}$$

From the definition of the reduced reciprocal system (113) and the two Eqs. (114) and (27), we obtain the following:

$$\begin{aligned} \hat{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = (\bar{A})^{-1} \\ \hat{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) = (\bar{A})^{-1}\bar{B} \\ \hat{C}_1 &= (C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = -\bar{C}(\bar{A})^{-1} \\ \hat{D} &= \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B} \end{aligned} \tag{116}$$

In virtue of Eq. (116), we have the following relationship between the two transfer functions $\bar{G}(s)$ and $\hat{G}_r(s)$ and written as:

$$\begin{aligned} \bar{G}(s) &= \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} + \bar{D} \\ &= \bar{C}\left(\frac{1}{s}\right)\left(I - \frac{1}{s}\bar{A}\right)^{-1}\bar{B} + \bar{D} \\ &= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1}\bar{A} - \frac{1}{s}\bar{A}\right)^{-1}\bar{B} + \bar{D} \\ &= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1} - \frac{I}{s}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \\ &= -\bar{C}\left(\frac{I}{s} - (\bar{A})^{-1} + \bar{A}\right)\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \\ &= -\bar{C}(\bar{A})^{-1}\bar{B} - \bar{C}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \\ &= -\bar{C}(\bar{A})^{-1}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B} \\ &= \hat{C}_1\left(\frac{I}{s} - \hat{A}_{11}\right)^{-1}\hat{B}_1 + \hat{D} \\ &= \hat{G}_r\left(\frac{1}{s}\right) \end{aligned} \tag{117}$$

Since the full system (A, B, C, D) is balanced and asymptotically stable and we have the balanced Gramian $\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$, we introduce the following theorem for balancing of the reduced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$.

Theorem B.1. *[21] The reduced order model $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ by singular perturbation approximation is balanced with Σ_1 and asymptotically stable.*

Proof. We know from Lemma A.4 that the reduced system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced with Σ_1 which satisfy the Lyapunov equations

$$\hat{A}_{11}\Sigma_1 + \Sigma_1\hat{A}_{11}^T + \hat{B}_1\hat{B}_1^T = 0$$
$$\hat{A}_{11}^T\Sigma_1 + \Sigma_1\hat{A}_{11} + \hat{C}_1^T\hat{C}_1 = 0$$

we multiply the first equation from the right by $\hat{A}_{11}^{-1}$ and from the left by $\hat{A}_{11}^{-T}$ to get

$$\hat{A}_{11}^{-1}(\hat{A}_{11}\Sigma_1)\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}(\Sigma_1\hat{A}_{11}^T)\hat{A}_{11}^{-T} + A^{-1}(\hat{B}_1\hat{B}_1^T)A^{-T} = 0$$
$$\Sigma_1\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}\Sigma_1 + (\hat{A}_{11}^{-1}\hat{B}_1)(\hat{A}_{11}^{-1}\hat{B}_1)^T = 0$$

substitute these values into Eq. (115) we obtain

$$\bar{A}\Sigma_1 + \Sigma_1\bar{A}^T + \bar{B}\bar{B}^T = 0$$

If the second Lyapunov equation is multiplied by $\hat{A}_{11}^{-T}$ from the right and by $\hat{A}_{11}^{-1}$ from the left, then we get

$$\hat{A}_{11}^{-T}(\hat{A}_{11}^T\Sigma_1)\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\Sigma_1\hat{A}_{11})\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\hat{C}_1^T\hat{C}_1)\hat{A}_{11}^{-1} = 0$$
$$\Sigma_1\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}\Sigma_1 + (\hat{C}_1\hat{A}_{11}^{-1})^T(\hat{C}_1\hat{A}_{11}^{-1}) = 0$$

In the same way from Eq. (115), we have

$$\bar{A}^T\Sigma_1 + \Sigma_1\bar{A} + \bar{C}^T\bar{C} = 0$$

Finally, our reduced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is balanced with Gramian Σ_1.

Since $\hat{A}_{11}$ is stable .i.e., $\Re\{\lambda(\hat{A}_{11})\} < 0$, where λ is an eigenvalue of $\hat{A}_{11}$, then the corresponding eigenvalue of $\bar{A}$ is $\frac{1}{\lambda}$ so we have $\mathbb{R}\{\lambda_i(\hat{A}_{11})\} < 0$ which mean the reduced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is asymptotically stable □

If the hypothesis of Theorem B.1 holds true, then there is an error bound available for the singular perturbation approximation $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of the stable and balanced system (A, B, C, D).

In the form of the H_∞ norm, the error bound is given as [15]:

$$\|G - \bar{G}_r\|_\infty \le 2\sum_{r=i+1}^{n}\sigma_i \tag{118}$$

Proof. From Eqs. (110), (117) and Lemma A.6, we have

$$\begin{aligned}
\|G(s) - \bar{G}(s)\|_\infty &= \|G(s) - \hat{G}(\tfrac{1}{s}) + \hat{G}(\tfrac{1}{s}) - \hat{G}_r(\tfrac{1}{s}) + \hat{G}_r(\tfrac{1}{s}) - \bar{G}(s)\|_\infty \\
&\le \|G(s) - \hat{G}(\tfrac{1}{s})\|_\infty + \|\hat{G}(\tfrac{1}{s}) - \hat{G}_r(\tfrac{1}{s})\|_\infty + \|\hat{G}_r(\tfrac{1}{s}) - \bar{G}(s)\|_\infty \\
&\le \|\hat{G}(\tfrac{1}{s}) - \hat{G}_r(\tfrac{1}{s})\|_\infty \\
&\le 2\sum_{i=r+1}^{n}\sigma_i
\end{aligned}$$

□

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